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JEL Classification: E31, E40, E50, E60

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A Toolkit for Solving Models with a Lower Bound on Interest Rates of Stochastic Duration

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Abstract

This paper presents a toolkit to solve for equilibrium in economies with the effective lower bound (ELB) on the nominal interest rate in a computationally efficient way under a special assumption about the underlying shock process, a two-state Markov process with an absorbing state. We illustrate the algorithm in the canonical New Keynesian model, replicating the optimal monetary policy in Eggertsson and Woodford (2003), as well as showing how the toolkit can be used to analyze the medium scale DSGE model developed by the Federal Reserve Bank of New York. As an application, we show how well various policy rules perform relative to the optimal commitment equilibrium. A key conclusion is that previously suggested policy rules – such as price level targeting and nominal GDP targeting – do not perform well when there is a small drop in the price level, as observed during the Great Recession, because they do not imply sufficiently strong commitment to low future interest rates (“make-up strategy”). We propose two new policy rules, Cumulative Nominal GDP Targeting Rule and Symmetric Dual-Objective Targeting Rule that are more robust.

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1 Introduction

The effective lower bound (ELB) on nominal interest rates has been widely studied in recent years. It is standard to analyze this problem with dynamic stochastic general equilibrium (DSGE) models, where the ELB shows up as an inequality constraint on the nominal interest rate. However, inequality constraints complicate the application of standard solution strategies, e.g. perturbation methods. These methods approximate the behavior of a dynamical non-linear model around a point (usually, but not necessarily, via linearization) using differentiability assumptions. Occasionally binding constraints pose a challenge for direct application of these methods.

In this paper, we present a toolkit aiming to facilitate the application of a generalized version of the solution method first used in Eggertsson and Woodford (2003), who analyze the ELB in the face of a two-state Markov process for the exogenous shocks with an absorbing state. We illustrate the algorithm in the canonical New Keynesian (NK) model and in the medium-scale DSGE model developed by the Federal Reserve Bank of New York (FRBNY). As an economic application, we consider various policy rules and study their performance relative to the optimal commitment equilibrium. Previously suggested policy rules – such as price level targeting and nominal GDP targeting – do not perform well when the price level does not fall by a large amount, as observed during the Great Recession, because they do not imply sufficiently strong commitment to low future interest rate (make-up strategy). To solve this shortcoming, we propose two new policy rules, Cumulative Nominal GDP Targeting Rule and Symmetric Dual-Objective Targeting Rule that are more robust.

Several strategies have been proposed to deal with the presence of inequality constraints in DSGE models. Eggertsson and Woodford (2003) exploit a particular structure for the exogenous disturbances: the shock process implies that the model unexpectedly moves to a “crisis state” and then reverts back to the “steady state” with a fixed probability. Once back to the steady state, it stays there forever (i.e. the steady state is an absorbing state). The idea behind the approach is intuitive: instead of treating a single dynamical system that contains both a set of equality constraints and a set of occasionally binding inequality constraints, we split the system into several parts, called “regimes”, each of which contains equality constraints exclusively. Once cast in this form, we can apply the standard perturbation machinery again.

An application to the ELB scenario should make this clear: we distinguish among four regimes, each of them corresponding to a different combination of the status of the inequality constraint (ELB binding or not) and the exogenous Markov disturbance (crisis or steady state). For the regimes that feature the ELB not binding, we treat the model as if the ELB was not present. In the other two regimes, when the ELB constraint is binding, the equilibrium conditions will be characterized by the equality constraint (e.g. $i_t = i_{t-1} = 0$). Since all four dynamical systems are characterized by a set of equations, each can be solved using perturbation techniques.

The assumptions on the shock structure allow us to solve the model recursively in regimes. We get a piecewise solution by starting with the last regime, where ELB is not relevant, and work backwards to the period when the shock firstly hits the system. Note that since outcomes in later regimes influence behavior in earlier ones through expectation formation, the strategy is not based on a simple merger of separate models and sticking their solutions together.

There are two key advantages to our approach: first, its relative simplicity allows for handling of models with many state variables; second, compared to competing local solution techniques, our strategy allows for a fully stochastic structure.
The toolkit features an algorithm that generalizes the solution method in Eggertsson and Woodford (2003). It allows for the case of a regime in which the two-state Markov process is in the crisis state, but the ELB is not binding. This feature is of particular importance for our application of analyzing policy rules; a common property of policy rules is that they imply an inertial response of the interest rate. An example would be a Taylor-type rule with lagged terms for the nominal interest rates. Rules of this kind often do not imply an immediate reduction of the interest rate to the ELB once the two-state Markov shock switches to the low state. The new feature is thus a meaningful addition and facilitates the analysis of different types of policy rules in the presence of an ELB, which is our main application in this paper.

The idea of attacking the problem by constructing a piecewise solution is not new, nor is the idea of a toolkit applying it. In fact, Guerrieri and Iacoviello (2015), henceforth OccBin, provide a toolkit for solving dynamic models with occasionally binding constraints in a similar fashion. Therefore we want to delineate the distinctive features of our approach. The main difference from OccBin is that we do not assume perfect foresight, i.e. a deterministic setting (this feature also differentiates our approach from several other strategies, such as Extended Path algorithm). To achieve this, we rely on the specific shock structure, a two-state Markov process with an absorbing state. Expectations about the future path of variables are a crucial component of models related to the ELB (e.g. uncertainty whether the economy will hit the ELB and uncertain timing of liftoff), and hence allowing for uncertainty is a useful feature of our toolkit.

Adding a two-state Markov process with an absorbing state usually implies the following timing for the models analyzed with the toolkit: initially, a shock hits the economy; the response of the central bank may or may not be to immediately lower the interest rate to zero. In every period there is some probability that the shock reverts to its initial condition. Once the shock reverts to the initial state, it stays there forever. There will often be a transition period, lasting from the point when the shock reverts to its initial level until all other variables of the model return to their steady state value. One benefit of our setup is that one can separately calibrate the expected duration of the constraint being binding from its actual, realized duration. Empirical evidence on the Great Recession, for instance the Blue Chip financial forecasts (Aspen Publishers 2008-12), hints that market participants were expecting the ELB to bind for a much shorter time than it turned out to be the case. Within our toolkit we can account for this evidence, and can analyze several questions related to it.

The expected duration of the ELB episode is not necessarily exogenously determined simply by the transition probability of the shock; in the case of a central bank that has commitment power, the duration of a binding ELB will typically be longer than the duration of the shock in its crisis state. The periods where the inequality constraint is binding therefore does not coincide with periods where the shock is in the low state. This means that the duration of the ELB episode will be endogenously determined in the model, depending on the optimal decisions taken by the monetary authority.

Our main application is monetary policy at the ELB. Since the standard policy tool of affecting nominal interest rates is not available anymore, influencing expectations about their future path becomes the main lever through which the monetary authority can affect present variables. In this environment, policy rules that are able to mimic some form of commitment from the central bank are believed to perform relatively well. For example, in Eggertsson and Woodford (2003), who predict a strong deflation, rules that commit to bringing the price level back to pre-crisis levels, and to inflate in the future, are very effective. We use the toolkit to show that previously proposed candidate rules, such as price level targeting and nominal GDP targeting, do not perform well in the absence of strong price movements.
We propose two new rules, a Cumulative Nominal GDP Targeting Rule and a Symmetric Dual-Objective Targeting Rule, that imply a commitment from the central bank to make up for past deviations from target on both the price level and output. We study their performance, in the standard NK model as well as in the NYFRB DSGE model, by comparing them to a set of rules that are standard in the literature. We do so in an environment with low inflation and small movement in the price level, as experienced during the Great Recession. Since both rules imply an aggressive reaction to past output misses, they manage to communicate that the longer the crisis, the more accommodative monetary policy will be. This in turn generates enough stimulus to prevent a large recession to start with.

Of previously proposed policy rules, the ones that perform best are the Superinertial Rule proposed by Rotemberg and Woodford (1999) and the Augmented Taylor Rule by Reifschneider and Williams (2000). Policy rules that perform relatively poorly, in addition to the already mentioned price level and nominal GDP targeting, include average inflation targeting, a result in line with Reifschneider and Wilcox (2019).

The paper is structured as follows: Section 2 outlines how the solution method relates to the literature; Section 3 presents the solution algorithm; Section 4 provides a few applications in the context previously defined; Section 5 concludes.

2 Related Literature

There exists a sizable literature on solution methods for DSGE models. Solution strategies can be classified into local and global methods. The former includes perturbation methods, the latter projection methods. Projection methods can handle occasionally binding constraints in a direct way, but they are associated with considerable computational burden and suffer from the curse of dimensionality.

Our approach relies on perturbation. Generally, provided the system in question is well behaved, perturbation methods can handle large models with many state variables and provide a high quality of approximation. However, in the presence of occasionally binding constraints differentiability does not hold everywhere.

Several attempts have been made to address this issue: our approach is a re-work and extension of the algorithm proposed in Eggertsson and Woodford (2003) that solves a NK model with a ELB in a fully stochastic setting. Jung, Teranishi and Watanabe (2005) is another early strategy based on a piecewise solution, but in a deterministic setting. The paper closest in spirit to ours is Guerrieri and Iacoviello (2015), already mentioned in the introduction. They develop a toolkit for solving deterministic dynamic models with occasionally binding constraints by piecewise first-order perturbation. The toolkit is not simply tailored for a specific model, but it is applicable to a general class of models and compatible with Dynare, a widely used software for solving and estimating rational expectations models. This approach comes with several benefits: it can handle an AR(1) shock structure, independent shocks, several alternations between slack and binding modes in one simulation. However, the toolkit assumes perfect foresight. From this discussion it should be clear that our toolkit is complementary to OccBin. Each is tailored to a particular set of questions. Both have in common that they are easily implementable.

Doubts have been raised whether linearization offers a good approximation to the fully non-linear system dynamics in the presence of the ELB, see for example Boneva, Braun and Waki (2016). Eggertsson and Singh (2019) look at the non-linear version of Eggertsson and Woodford (2003) and find that the

\textsuperscript{1}For an overview we refer to the recent handbook chapter by Fernández-Villaverde, Rubio-Ramrez and Schorfheide (2016).
approximation error is modest even for large shocks. They also comment on poor approximation results that have been mentioned in relation to Rotemberg pricing, a finding they ultimately trace to particular assumptions rather than to errors introduced by linearization.

One way to account for the ELB is to replace the inequality constraint with news shocks: ones that are realized some time before they actually enter equations of the model. Lasen and Svensson (2011) and Holden and Paetz (2012) develop this point: they use such shocks to transform a non-linear constraint into ”as if” linear models. The approach is able to handle higher order approximations as well as uncertainty. Holden (2016) offers a Dynare toolbox for the easy implementation of the procedure, DynareOCB, providing reliable accuracy with sufficient speed.

Another local method of solving large systems within reasonable time is the Extended Path algorithm (EP), proposed by Fair and Taylor (1983) and applied to a model with a ELB by Adjemian and Juillard (2011). This method sets a terminal date when the solution trajectory is assumed to be sufficiently close to the steady state. The EP algorithm is solved under perfect foresight: at each point in time, all future shocks are assumed to be equal to their expected values of zero. Adjemian and Juillard (2013) note that ignoring Jensen’s inequality under an EP approach leads to a sizable approximation error in models with occasionally-binding constraints. The authors extend EP to a Stochastic Extended Path algorithm that is more suitable for a setting with non-linear constraints. It is somewhat of a middle ground between perfect foresight and a fully stochastic setting: for some finite number of periods ahead, the setting is stochastic (e.g. the expectation is explicitly approximated via quadrature integration) while after that period all shocks are assumed to be zero.

Another strategy is to replace the inequality constraint by a smooth penalty function, thereby eliminating the inequality constraints from the model (this is also referred to as barrier method). The idea is to penalize the agents’ utility in cases where the inequality constraint is violated. This method is outlined in Judd (1998) and put to use in Preston and Roca (2007) and Kim, Kollmann and Kim (2010), among others. The advantage of the approaches discussed so far is that they can manage medium to large sized models in reasonable time. Global solution methods can account for non-linearities and deliver solutions with high precision, but suffer from the curse of dimensionality.

Some papers using projection methods to analyze the ELB should also be mentioned here: Adam and Billi (2006) and Nakov (2008) work with the linearized system of equations of standard NK models, and the ELB is the only source of non-linearity. Fernández-Villaverde, Rubio-Ramrez and Schorfheide (2016) solve the fully non-linear model and highlight the benefits of this approach.

Given the numerous and powerful alternatives available, we do not want to claim superiority to any of the presented methods. Which approach suits best for a particular problem at hand will have to be evaluated on an individual basis. Our method has the advantage of tractability, simplicity, fast implementation and the flexibility to account for a particular kind of uncertainty that has been popular in the literature.

In an application of the toolkit, we analyze optimal policy rules in economies at risk of hitting the ELB. There is a vast literature analyzing optimal rules. Recent contributions include Kiley and Roberts (2017), Mertens and Williams (2019a) and Mertens and Williams (2019b). Nakov (2008) is an earlier paper looking at optimal rules at the ELB. Our analysis highlights the implications of a stable price level for...
rules like price level targeting and nominal GDP targeting, and proposes rules that are robust to such environments.

3 A Piecewise Linear Solution to Rational Expectations Models with Inequality Constraints

3.1 Basic Idea

In this section we outline our approach in applying perturbation methods to models with inequality constraints. Technically, the use of the implicit function theorem (IFT) on which perturbation methods rely requires the function approximated to be smooth. Inequality constraints introduce a kink in the model at the point where the inequality constraint becomes binding. On a more intuitive level, once the inequality constraint becomes binding (or slack, depending on the point of approximation), local approximation via IFT becomes inaccurate.

The basic idea of our approach is to circumvent this problem by the following strategy: we split the model into several parts, called regimes, depending on which set of equations applies. This strategy decomposes the single inequality constraint that characterizes the model in general into several equality constraints that characterize each regime separately. This turns one system with occasionally binding constraints into several interlinked systems. Critically though, it does not simply amount to transforming the problem into independent dynamical linear systems: optimal decision-making in the current regime takes into account uncertainty as to what regime will govern future dynamics of the model, thus linking the dynamics of one regime to the next.

We assume the fundamental shock $\epsilon_t \in \{\epsilon_L, \epsilon_H\}$ hitting the system to be characterized by a two-state Markov process with an absorbing state, i.e. $P(\epsilon_{t+1} = \epsilon_L | \epsilon_t = \epsilon_L) = \mu$ and $P(\epsilon_{t+1} = \epsilon_H | \epsilon_t = \epsilon_H) = 1$. This assumption buys a lot: this shock structure allows for the decomposition of a single dynamical system into several regimes.

The regimes are:

0. $0 < t < \tilde{T}$: the shock is in the low state, $\epsilon_t = \epsilon_L$ and the inequality constraint is not yet binding;

1. $\tilde{T} < t < \tau$: the shock is in the low state, $\epsilon_t = \epsilon_L$, and the inequality constraint is binding;

2. $\tau < t < \tau + k\tau$: the shock has returned to the high, absorbing state, $\epsilon_t = \epsilon_H$ but the inequality constraint is binding;

3. $\tau + k\tau \leq t$: the shock remains in the absorbing state, $\epsilon_t = \epsilon_H$, and the inequality constraint is not binding.

---


6 The exogenous shock $\epsilon_t$ needs not to be a scalar. As long as the shocks are perfectly correlated, one should think of $\epsilon_t$ as a vector of perfectly correlated shocks.

7 An example of this in the context of the ELB would be optimal policy by the central bank under commitment. Here we can split the problem into 3 regimes, one where the shock is over and the ELB is slack, one where the shock is over but the ELB binds, and one where the shock is on an the ELB binds. Note that not every problem needs to have three (or four) regimes: the problem of the central bank under discretion (without endogenous state variables) will typically only have two regimes.

8 This regime generalizes the solution method in Eggertsson and Woodford (2003) to allow for very sluggish policy rules. Notice that $\tilde{T}$ can be 1, meaning that regime 0 never starts. This is the case, for example, of a fully forward-looking economy such as the standard NK model with a policy rule that has no lagged term.

9 Notice that the toolkit allows for $\tilde{T}$ to be “large enough”, meaning that regime 1 never starts. This generalizes the use of the toolkit for shocks that are enough small to not trigger the inequality constraint to bind.
At time $t = 1$ the shock admits the low state and hits the system, $\epsilon_1 = \epsilon_L$. In $t = \tau$ the shock reverts to the high, absorbing state and stays there forever.

As for the inequality constraint, there are a few switching points: first, the inequality constraint is binding starting from some point in time $\tilde{T}$; this is the switching time from regime 0 to regime 1. Note that from the agents’ perspectives, $\tilde{T}$ is known and deterministic.$^{10}$ Second, when the shock has reverted to its absorbing state at time $\tau$ the inequality constraint could still be binding. This situation constitutes a separate regime and the switching time is stochastic by nature as it is governed by the Markov process. $\tau$ is the switching time from regime 0 or 1 to regime 2. The duration of regime 2 is denoted by $k$, and might depend on the duration of the preceding regimes. Finally, $\tau + k\tau$ represents the time when the economy switches from regime 2 to regime 3, where the shock is in its absorbing state and the constraint does not bind anymore; regime 3 governs the dynamics thereafter.

To take into account the expectations channel that interlinks the regimes, we solve the problem backwards, starting with with regime 3. Note that since we assume an absorbing state for the exogenous shock, the system is deterministic in regimes 2 and 3. It is in regimes 0 and 1, where expectations about future realizations of the shock affect the current behavior, that the stochastic setting comes into play.

### 3.2 General Formulation

To introduce technical notation, many macroeconomic models can be formulated in the following form:

$$ E_t \tilde{f} (\tilde{\xi}_{t+1}, \tilde{\xi}_t, \tilde{\epsilon}_t) \leq 0 \quad (1) $$

where $\tilde{\xi}_t$ is a vector of endogenous variables and $\tilde{\epsilon}_t$ is a (vector) shock following a two-state Markov process with an absorbing state.$^{11}$ The shock switches back to the absorbing state with probability one at a (deterministic) time, $\tau_{\text{max}}$. $E_t$ is the mathematical operator for expectation conditional on information available at time $t$ and $\tilde{f} (\cdot, \cdot, \cdot)$ is a differentiable function. Typically, $\tilde{f}$ contains structural equations and/or necessary first order conditions of an optimization problem arising in a microfounded economic model. The inequality sign in (1) accounts for the presence of inequality constraints.$^{13}$ For notational convenience, we transform the notation in (1) as follows:

$$ E_t f^* (\xi^*_{t+1}, \xi^*_t) \leq 0 \quad (2) $$

where $\xi^*_t = [\tilde{\xi}_t, \tilde{\epsilon}_{t-1}]^T = [Z^*_t, S^*_{t-1}, \tilde{\epsilon}_{t-1}]^T$ is a vector of $NY^* = (NZ + NS + Ne)$ elements, of which $NZ$ non-predicted (or jump) variables $Z^*_t$, $NS$ predetermined variables $S^*_{t-1}$, and $Ne$ exogenous disturbances $\tilde{\epsilon}_{t-1}$.

In order to deal with the inequality constraints, we propose to split the problem into several regimes. In each regime the system of equations can take on a different form, so we denote each resulting system by another subscript. Importantly, the inequality constraints are either slack or binding, so that we write a

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$^{10}$While the point in time $\tilde{T}$ is known and deterministic, a realization of the shock $\tau < \tilde{T}$ will in most cases mean that the ELB does not bind at all. Thus ex ante, there can be a positive probability on the event that a shock realization will materialize such that the ELB does not bind at all.

$^{11}$This notation requires that exogenous variables appear in present values only.

$^{12}$Note that $\tau_{\text{max}}$ can be set so that the period at which the shock reverts to its high state is arbitrarily far in the future.

$^{13}$The full set of first order conditions usually also includes transversality conditions, No-Ponzi scheme conditions and initial conditions. For brevity’s sake we do not explicitly formulate them. Moreover, note that this formulation is general enough to include higher order difference equations, since they can be easily transformed into first order difference equations by redefining $\tilde{\xi}$.
system of equations instead of inequalities. We denote each system of equations by the following:

\[ E_t f^*_i (\xi^*_t, \xi_t) = 0 \] (3)

where \( i \in \{0, 1, 2, 3\} \) indicates the regime.

Next, we approximate the non-linear systems of equations in (3) to the first order around a specific point. Usually, we approximate around the deterministic steady state \( \bar{\xi} \) of some “baseline regime” \( \tilde{i} \) that fulfills:

\[ f^*_i (\bar{\xi}, \bar{\xi}) = 0 \] (4)

We can choose to linearize around an equilibrium point of any regime, and the particular choice will depend on the application. We can write the resulting linear system of equations in the following general form:

\[ A^i E_i \xi_{t+1} = B^i \xi_t \] (5)

where \( A^i, B^i \) are \( NY \times NY \) coefficient matrices and \( \xi_t = [Z_t, S_{t-1}, e_{t-1}, C_t]' = [Z_t, P_{t-1}]' \) is a \( NY = (NY^* + 1) \times 1 \) vector.\(^{14}\) The variables \( \xi_t, Z_T, S_t, e_t, P_t \) are now written without star: this due to the process of linearization, where the variables are often transformed, e.g. to log-deviation from steady state. The term \( C_t \) accounts for binding inequality constraint.\(^{15}\)

The solution path to small perturbations that is derived from the linearized system in (5) will be exact only up to a residual of order \( O(||\epsilon, \delta||^2) \). The \( O(.) \) term is the remainder appearing in the first-order Taylor-series approximation. In addition, \( \epsilon \) in the big-O expression refers to the perturbation of the shock variable and \( \delta \) takes account of the fact that the equilibrium in regimes other than the baseline might not coincide with the point we linearize around. This amounts to another “perturbation” that brings us off the point we approximate around. In our examples we will usually have \( \delta = \frac{i - \bar{i}}{1 + \gamma} \), with \( \bar{i} \) being the non-stochastic steady-state interest rate. Then we will have \( \delta \neq 0 \) if \( i = 0 \), which is the case in regimes 1 and 2.

As noted in Eggertsson and Woodford (2003) – as well as in Woodford (2003) p.383 ff. – perturbation theory provides accurate linearized solutions only for shocks that are small enough, i.e. both \( \epsilon \) and \( \delta \) small. We now briefly discuss this assumption: Eggertsson and Woodford (2003) show that we can make \( \delta \) arbitrarily small by assuming that there is interest paid on base money. In our application, \( \delta \) is close to realistic values (e.g. a drop of the interest rate from 3% to 0%), and we hope (without verifying) that our linear approximation will be accurate. Eggertsson and Singh (2019) compare the fully non-linear version of a standard NK model with a binding ELB to the log-linearized one, and find that they behave similarly for reasonable values of disturbances, including ours.

The system in (5) has a familiar form and we can therefore apply standard rational-expectations solution methods. All regimes except the last will have a finite duration. Blanchard and Kahn (1980) present the conditions under which a system of infinite horizon like (5) has a unique bounded solution. The regimes

\(^{14}\)The vector \( P_t \) contains all predetermined variables, which include past exogenous shocks as well as constants.

\(^{15}\)When the constraint is not binding we typically have \( C_t = 0 \). To give a concrete example, consider the zero lower bound and the nominal rate \( i_t \). We can define \( i_t = \frac{\bar{i} - \bar{i}_0}{1 + \gamma} \). When the zero bound is binding, we have \( i_t = 0 \) and \( \bar{i} = i = -\frac{1}{1 + \gamma} \). This is an example of a constant term.
before the last will have a finite terminal date and can be solved recursively. Solutions will take the form:

\[ P_t = G^*_t P_{t-1} \]  
\[ Z_t = D^*_t P_t \]  

(6)  
(7)

where \( G^*_t \) and \( D^*_t \) are transition matrices. We see that time \( t \) variables will be a linear combination of pre-determined variables, shocks and constants. Once we find all transition matrices an initial condition for the predetermined variables \( P_0 \) allows us to compute the evolution of the dynamical system.

3.3 The Solution Algorithm

After this general outline of our approach, we want to show the way to derive the transition matrices of every regime. Since transition matrices of earlier regimes depend on later ones, we solve the problem recursively. We will first take the length of each regime as given and then discuss how to determine a vector \( k \) and the time \( \tilde{T} \). The former contains the length of regime 2 for each realization of the shock path. We will refer to those realizations as *contingencies*.

3.3.1 Finding the transition matrices in all four regimes given \( k \) and \( \tilde{T} \)

Here we describe the construction of matrices \( G^*_t \) and \( D^*_t \) for each regime. We will rename the matrices according to the following notation, so that they correspond to the state they are in.

\[
\begin{array}{c|cccccccc}
\text{Period} & t < \tilde{T} & \tilde{T} & t < \tau & \tau & \tau \leq t < \tau + k_{\tau} & t \geq \tau + k_{\tau} \\
\hline
D_t & D^0_t & \ldots & D^1_t & \ldots & D^2_{k_{\tau}} & \ldots & D^2_j & \ldots & D^3 \\
G_t & G^0_t & \ldots & G^1_t & \ldots & G^2_{k_{\tau}} & \ldots & G^2_j & \ldots & G^3
\end{array}
\]

where \( j = k_{\tau} - (t - \tau) \) denotes how many periods are left until the ELB is no longer binding.

**Solution for regime 3:** \( t \geq \tau + k_{\tau} \). The system can be written in the following form:

\[
A^3 E_t \begin{bmatrix} Z_{t+1} \\ P_t \end{bmatrix} = B^3 \begin{bmatrix} Z_t \\ P_{t-1} \end{bmatrix}
\]

(8)

A system of this form can be solved by the method outlined in Blanchard and Kahn (1980) as well as in King and Watson (1998).

The solution takes the form:

\[
P_t = G^3 P_{t-1} \\
Z_t = D^3 P_{t-1}
\]

Superscript 3 denotes the regime, and no subscripts are present because \( D^3 \) and \( G^3 \) do not depend on time.

**Solution for regime 2:** \( \tau \leq t < \tau + k_{\tau} \). As in the previous case, the system can be written as:
\[
A^2 E_t \begin{bmatrix} Z_{t+1} \\ P_t \end{bmatrix} = B^2 \begin{bmatrix} Z_t \\ P_{t-1} \end{bmatrix} 
\]

(9)

\[
\begin{bmatrix} A_1^2 & A_2^2 \\ A_3^2 & A_4^2 \end{bmatrix} \begin{bmatrix} E_t Z_{t+1} \\ P_t \end{bmatrix} = \begin{bmatrix} B_1^2 & B_2^2 \\ B_3^2 & B_4^2 \end{bmatrix} \begin{bmatrix} Z_t \\ P_{t-1} \end{bmatrix}
\]

and we want to find a solution of the form:

\[
P_t = G^{2j} P_{t-1}
\]

\[
Z_t = D^{2j} P_{t-1}
\]

We show how to derive those matrices in the following Lemma 1.

**Lemma 1.** Let \( \tilde{A} \) be the reduced row echelon form of \( \tilde{B} \), where

\[
\tilde{A} = \begin{bmatrix} I & 0 & -C_1 & -C_2 \\ 0 & I & -C_3 & -C_4 \end{bmatrix} \quad \text{and} \quad \tilde{B} = \begin{bmatrix} A_2^2 & -B_1^2 & -B_2^2 & A_1^2 \\ A_4^2 & -B_3^2 & -B_4^2 & A_3^2 \end{bmatrix}
\]

The solution in regime 2 will be:\(^{16}\)

\[
G^{2j} = \left[I - C_2 D^{2j-1}\right]^{-1} C_1
\]

\[
D^{2j} = C_3 + C_4 D^{2j-1} G^{2j}
\]

Also, it will hold \( D^{2,0} = D^3 \), and \( G^{2,0} = G^3 \).

**Proof.** See Appendix A.1. \( \square \)

**Solution for regime 1:** \( t < \tau \). Similarly to regime 2, we can write the solution into the form

\[
\begin{bmatrix} A_1^1 & A_2^1 \\ A_3^1 & A_4^1 \end{bmatrix} \begin{bmatrix} E_t Z_{t+1} \\ P_t \end{bmatrix} = \begin{bmatrix} B_1^1 & B_2^1 \\ B_3^1 & B_4^1 \end{bmatrix} \begin{bmatrix} Z_t \\ P_{t-1} \end{bmatrix}
\]

Note that the transition matrices will now be time varying. This is because in regime 1 the expectations at time \( t \) depend on \( k_{t+1} \), since \( E_t Z_{t+1} = \mu Z_{t+1|\epsilon_{t+1}=\epsilon_t} + (1-\mu) Z_{t+1|\epsilon_{t+1}=\epsilon_{t+1}} = \mu D_{t+1}^{1} P_t + (1-\mu) D^{2k_{t+1}} P_t = \tilde{D}_t^1 P_t \). Recall that with probability \( \mu \) the shock will stay in the low state, while with probability \( 1-\mu \) the shock will switch back to the high state and the coefficient matrix \( D \) will be calculated according to the next regime.

Let \( \tilde{C}_t \) be the row reduced echelon form of \( \tilde{D}_t \), where:

\[
\tilde{C}_t = \begin{bmatrix} I & 0 & -C_{t,1} & -C_{t,2} \\ 0 & I & -C_{t,3} & -C_{t,4} \end{bmatrix} \quad \text{and} \quad \tilde{D}_t = \begin{bmatrix} A_{t,2} & -B_{t,1} & -B_{t,2} & A_{t,1} \\ A_{t,4} & -B_{t,3} & -B_{t,4} & A_{t,3} \end{bmatrix}
\]

\(^{16}\)We make use of the deterministic nature of regimes 2 and 3 to ignore the expectation term.
Then, we can rewrite the system to be:

\[
\begin{bmatrix}
P_t \\
Z_t
\end{bmatrix} = \begin{bmatrix} C_{t,1} & C_{t,2} \\
C_{t,3} & \tilde{C}_{t,4} \\
\end{bmatrix} \begin{bmatrix} P_{t-1} \\
E_{t} Z_{t+1}
\end{bmatrix} = \begin{bmatrix} C_{t,1} & C_{t,2} \\
C_{t,3} & \tilde{C}_{t,4} \\
\end{bmatrix} \begin{bmatrix} P_{t-1} \\
\tilde{D}_{1} P_{t}
\end{bmatrix}
\] (11)

Once again, we want to find a solution of the form:

\[
P_t = G_{1,t} P_{t-1} \\
Z_t = D_{1,t} P_{t-1}
\]

To solve for these matrices we assume that at some time \( \tau_{\text{max}} \) the probability of the shock returning to its absorbing state (given that this did not happen at time \( \tau_{\text{max}} - 1 \)) is no longer \( 1 - \mu \), but 1. This allows us to use the same solution strategy as in regime 2 to obtain:

\[
G_{1,\tau_{\text{max}} - 1} = \left[ I - C_2 D_{2,k_{\text{max}}} \right]^{-1} C_{\tau_{\text{max}},1} \\
D_{1,\tau_{\text{max}} - 1} = C_3 + C_4 D_{2,k_{\text{max}}} G_{1,\tau_{\text{max}} - 1}
\]

We can apply this methodology recursively for \( t < \tau_{\text{max}} - 1 \):

\[
G_{1,t} = \left[ I - C_{t,2} \tilde{D}_{1,t} \right]^{-1} C_{t,1} \\
D_{1,t} = C_{t,3} + C_{t,4} \tilde{D}_{1,t} G_{1,t}
\]

**Solution for regime 0: \( t < \tilde{T} \).** The solution for regime 0 follows the same procedure to the one for regime 1. The only difference is that the matrices \( A \) and \( B \) differ from those in regime 1. Regime 0 holds until time \( t = \tilde{T} \). The transition matrices in regime 0 will have the form:

\[
G_{0,t} = \left[ I - C_{1,2} \tilde{D}_{1,t} \right]^{-1} C_{1,t,1} \\
D_{0,t} = C_{1,3} + C_{1,4} \tilde{D}_{1,t} G_{0,t}
\]

So far we have taken \( k \) (the duration of regime 2 for each and every \( \tau \)) and \( \tilde{T} \) (the period in which regime 1 starts holding) to be known by the agents. The next sections explain the algorithm we use to find them.

### 3.3.2 Finding \( k \) given \( \tilde{T} \)

The algorithm to find \( k \) for a given \( \tilde{T} \) is straightforward: we start from a vector of zeros, namely that regime 2 is never believed to hold, under any contingency. We then use the transition matrices found as described in the previous section and trace the evolution of all variables, including the one constrained with inequality (e.g. the nominal interest rate). Starting from the first contingency, we then check whether the inequality constraint is violated in the first period of regime 3. In case it is not violated, we move forward to the next contingency until we encounter a contingency for which – according to the transition matrices – the inequality is violated in the first period of regime 3. Whenever we find such a contingency, we update \( k \) by adding one unit to the duration of regime 2 under such contingency.

In the interest of saving computing power, it is possible to assume that there is an increasing relationship
between the contingency and the length of regime 2. Such an assumption is harmless in some models but it is still to be shown that the results are not affected in general. If one wants to verify that a given \( k \) does indeed imply a solution to the dynamic system, it is possible to check that the necessary condition on the variable whose constraint is potentially binding is not violated.\(^{17}\) After having updated \( k \), we proceed as explained in the previous sections by calculating a new set of transition matrices and then we follow the algorithm outlined above to check if there is some contingency for which the inequality is violated in the first period of regime 3.\(^{18}\)

### 3.3.3 Finding \( \tilde{T} \)

The algorithm to find \( T \) is intuitive. For a given \( T \), say \( T \) (which is initialized at 1), we take two steps to check whether it is the solution: first, for \( \tilde{T} = T \), meaning that regime 1 starts at time \( T \), we check that the inequality constraint is not violated at \( t = T - 1 \); second, we impose \( \tilde{T} = T + 1 \), assuming that regime 1 starts at \( T + 1 \). We find the corresponding \( k \), simulate the economy forward, and finally check whether the inequality constraint is violated for \( t = T \). If the inequality constraint is indeed violated, then regime 1 should start at \( t = T + 1 - 1 = T \) and we conclude that \( \tilde{T} = T \). If the inequality constraint is not violated for \( t = T \), this implies that regime 0 is correctly imposed in \( t = T + 1 \). In such case we continue iterating by increasing \( T \) by one.

### 4 Applications

As an application we revisit Eggertsson and Woodford (2003), henceforth EW2003, and analyze the optimal monetary policy at the ELB in the standard NK model. We then ask what kind of policy rule can implement it. Our key finding is that EW2003 suggested simplified price level targeting rule does poorly in replicating the optimal commitment in numerical experiments where the price level does not drop much at the ELB. We consider this scenario because of its similarity to the Great Recession in the United States, and because it stands in stark contrast with the experiment considered in EW2003. We also show that popular policy proposals such as nominal GDP targeting do relatively poorly in simulations in which there is not a significant fall in the price level at the ELB. We explain the logic of this result and suggest new alternative policy rules that do better. Finally, we confirm that the insights from the numerical experiments in the standard two-equation NK model extends to a medium scale quantitative model, such as the one used by the Federal Reserve Bank of New York.

#### 4.1 The Optimal Policy Commitment in Eggertsson and Woodford (2003)

EW2003 present the standard two-equation NK model and analyze the optimal monetary policy under commitment (OCP) taking account of the ELB. The policy can be represented by the following set of

\(^{17}\)This situation arises for example when a Taylor rule does prescribe a negative nominal interest rate, or – as in Eggertsson and Woodford (2003) – when the Lagrange multiplier is correctly non-negative.

\(^{18}\)Following this procedure, it is possible to find a \( k \) that solves the dynamic system with binding constraints. The existence of a \( k \) does not imply its uniqueness. However, we never found multiple solutions to the same problem when applying our solution method.
Figure 1: Optimal Commitment Policy as in Eggertsson and Woodford (2003). Colored lines are the impulse response for output ($\hat{Y}$), inflation ($\pi$), the nominal interest rate ($i$), and natural rate ($r^n$), gray lines represent the evolution for single contingencies (from 2 to 15). The vertical axes report deviations from steady state, in percentage points (annualized). The vertical axis for the interest rate reports annualized percentage points.
\[
\dot{Y}_t = E_t \dot{Y}_{t+1} - \sigma (i_t - E_t \dot{\pi}_t + r^n_t) \\
\dot{\pi}_t = \kappa \dot{Y}_t + \beta E_t \dot{\pi}_{t+1} \\
0 = \dot{\pi}_t + \phi_{1t} - \phi_{2t-1} - \frac{1}{\beta} \sigma \phi_{1t-1} \\
0 = \lambda \dot{Y}_t + \phi_{1t} - \frac{1}{\beta} \phi_{1t-1} - \kappa \phi_{2t} \\
\phi_{1t} \geq 0 \\
i_t \geq 0 \\
\phi_{1t} i_t = 0
\]

where \(\dot{Y}_t\) is output in deviation from steady state, \(\dot{\pi}_t\) is inflation, \(i_t\) is the nominal interest rate, \(\phi_{1t}\) and \(\phi_{2t}\) are Lagrange multipliers, and \(r^n_t\) is an exogenous disturbance – the natural rate of interest – that follows a two-state Markov process. Specifically, there is an unexpected reduction in the natural rate of interest in period 1 so that \(r^n_1 = r^L < 0\), and

\[
\begin{cases}
   r^L & \text{w.p. } \mu \quad \text{if } r^n_{t-1} = r^L \\
   r^H & \text{w.p. } 1 - \mu \quad \text{if } r^n_{t-1} = r^L \\
   r^H & \text{if } r^n_{t-1} = r^H 
\end{cases}
\]

The toolkit simulates the economy described above and produces the outputs as shown in Figure 1. We outline the steps required for coding the simulation in Appendix A.2. Each of the light gray lines represents a contingency, i.e. a specific realization of the Markov chain. Note that the time period at which the exogenous disturbance switches to its absorbing state is sufficient to characterize the specific realization. For this reason, one can refer to a contingency as the period at which the exogenous disturbance is back to its steady state. For example, the third gray line from the left for inflation, output and interest rates corresponds to the case in which the natural rate of interest reverts back to steady state in period 4. Observe that the evolution of a variable in a given contingency, prior to the shock reverting to steady state, depends on expectation about the evolution of variables in all future contingencies. The purple lines represent impulse response functions (IRF). Those are weighted averages of the evolution of each variable and correspond to the expectation agents hold about the economy in the initial period after the shock has been realized.

### 4.2 Implementing the Optimal Commitment in Eggertsson and Woodford (2003) via a Price Level Target and a Nominal GDP Target

In Figure 2 we compare the optimal commitment to the standard Taylor rule (TTR) using the numerical values assumed in EW2003. As emphasized by EW2003, the Taylor rule – or equivalently a strict inflation target – results in a large output drop (about 15 percent) and drop in inflation (about 10 percent) at the ELB. The OCP – via the central bank committing to keeping the nominal interest rate low for a substantial period of time – eliminates most of the drop in output. EW2003 show that a policy rule that fully implements this equilibrium can be described as follows: at the ELB the central bank commits to

\(^{19}\)See Eggertsson and Woodford (2003) for more details on the derivation. We use the EW2003 parametrization: \(\theta = 7.87, \sigma = 0.5, \kappa = 0.02, \beta = 0.99, \lambda = \frac{1}{\theta}, \mu = 0.9, r^L = -0.005, r^H = \beta^{-1} - 1.\)
Figure 2: Selected rules and optimal commitment (OCP) under Eggertsson and Woodford (2003) calibration. Lines represent contingency 10 for output ($\hat{Y}$), inflation ($\pi$), the nominal interest rate ($i$), and nominal GDP ($\hat{N}$). The vertical axes report deviations from steady state, in percentage points (annualized figures). The vertical axis for the nominal interest rate reports annualized percentage points.
not increasing the interest rates unless it reaches a certain threshold, defined by:

\[ \tilde{P}_t = \tilde{P}_t^* \] (20)

where \( \tilde{P}_t \) is the weighted average of (detrended) output and the (detrended) price index defined by:

\[ \tilde{P}_t \equiv \tilde{P}_t + \frac{K}{\lambda} \tilde{Y}_t. \] (21)

The key to this commitment is how the threshold \( \tilde{P}_t^* \) is formulated. EW2003 show that the optimal monetary policy commitment is replicated in the case \( \tilde{P}_t^* \) is computed according to the following formula:

\[ \tilde{P}_t^* = \tilde{P}_t^* + \beta^{-1} (1 + \kappa \sigma) \Delta_t - \beta^{-1} \Delta_{t-1} \] (22)

where \( \Delta_t \) is a variable that measures by how much the monetary authority misses its target in period \( t \) due the ELB:

\[ \Delta_t \equiv \tilde{P}_t - \tilde{P}_t^*. \] (23)

EW2003 recognize that this rule might be difficult to communicate in practice. Hence, they suggest the following simplified variation of the optimal rule:

\[ \hat{P}_t + \frac{K}{\lambda} \hat{Y}_t = \hat{P}^* \] (24)

Now, the gap-adjusted price level target is fixed at all times. Figure 2 shows that this simplified policy rule (PLT) does a relatively good job in replicating OCP. The reason is that the fall in the price level through the duration of the shock commits the central bank to a policy easing once the shock has subsided. Thus, the interest rate remains at its lower bound even once the shock has reverted, and inflation and output gap could be set at their steady state value. This is the key feature of the optimal monetary commitment. As stressed by Woodford (2012), targeting nominal GDP instead of the price level has the same essential features. This kind of policy has been suggested by a number of authors such as Hatzius and Stehn (2011) and Sumner (2012). A nominal GDP target can be written as:

\[ \hat{P}_t + \frac{K}{\lambda} \hat{Y}_t = \hat{Y}^* \] (25)

which would equivalent to the simplified price level target in EW2003 for special values of \( \lambda \). Figure 2 shows that this policy, denoted NGDPT, does a relatively good job in replicating the optimal commitment policy in the EW2003 numerical example. Again, the key is that the fall in the price level implies a substantial monetary easing even once the shock has reverted back to steady state, as mandated by the optimal policy commitment.

### 4.3 The Great Recession and the Robustness of Nominal GDP and a Price Level Target

The key take-away from the last Section was that the simple Price Level Target suggested in EW2003 and the Nominal GDP Target replicated the optimal commitment relatively well in the EW2003 numerical example. As we discussed, this is explained by the fact that the fall in the price level generates a commitment to lower future nominal interest rates once the shock has reverted to steady state, while a
standard Taylor rule would imply an immediate normalization at that time.

A key feature of the EW2003 calibration, however, is that if one assumes standard policy rules, such as the Taylor rule, there is substantial fall in the price level – of about 10 percent per year, as shown in Figure 2. Meanwhile, in the US Great Recession, the fall in the price level was much smaller by most accounts. Inflation, as measured by Personal Consumption Expenditure (PCE) for example, averaged at about 1.5 percent from 2008 to 2015, when the Fed started raising rates, which is only -0.5 percent below the 2 percent inflation target of the Fed. This is in sharp contrast to the 10 percent drop in inflation predicted by the EW2003 parametrization.

We now analyze the performance of the price level and the nominal GDP target policy rules once we calibrate the model to match a smaller drop in inflation. For this experiment, we interpret\(^{20}\) inflation as the deviation of inflation from target, as in Benigno, Eggertsson and Romei (2020).\(^{21}\)

To parameterize the model, we assume the same values for \(\kappa\) and \(\sigma\) as EW2003. Moreover, we assume the presence of a cost push shock\(^{22}\) \(u_t\) that is perfectly correlated with the natural rate of interest and thus follows the same two-state Markov process we have already discussed. We then choose both \(r_L\) as well as \(u_L\) in order to match a drop in inflation of -0.5 percent and output of -7.5 percent, thus taking on different values relative to the EW2003.\(^{23}\) A key assumption is that we suppose that policy was conducted according to the Taylor rule in this period. This calibration strategy results in \(r_L = -0.013875\) and \(u_L = 0.00136375\).

The idea behind the calibration is that the shock that gave rise to the Great Recession – for example a debt-deleveraging shock or a shock originating in the financial sector – simultaneously leads to a cost push shock and a drop in the natural rate of interest. While we model this in a reduced form, it is also the explanation given for the lack of deflation during the Great Recession in Eggertsson and Krugman (2012), who derive a fully specified microfoundation.\(^{24}\) Our approach is also consistent with the estimated DSGE model in Christiano, Eichenbaum and Trabandt (2015).

As an alternative to choosing a cost push shock to match the limited drop in inflation, we experiment with different values of \(\kappa\) that generate the small drop in inflation observed in the data. We show in Section 4.6 that our conclusions are robust to this alternative strategy.

We also change the EW2003 calibration in another important respect. The objective of the central bank in EW2003 is

\[
E_0 \sum_{t=0}^{\infty} \beta^t [\hat{\pi}_t^2 + \lambda \hat{Y}_t^2] \tag{26}
\]

A well-known feature of the standard NK model is that it places virtually no weight on output in the welfare objective of the government. In the EW2003 calibration, while the weight on the squared deviation of inflation from target is one, the weight on output is only \(\lambda = \kappa / \theta = 0.0025\). Here, instead, we assume equal weights on annual inflation and on output, so that \(\lambda = 1/16\).

The reason for making this alternative assumption is two-fold: first, the Federal Reserve typically

\(^{20}\)To stay consistent with the previous Section, we stick to the Eggertsson and Woodford (2003) calibration of \(\beta = 0.99\) and \(\bar{\pi} = 0\), which implies a steady state real and nominal rate of 4%. Numerical results are almost unchanged if we set \(\beta = 0.995\) and \(\bar{\pi} = 2\%\), implying a real rate of 2% and a nominal rate of 4%.

\(^{21}\)See Benigno, Eggertsson and Romei (2020) for the microfoundations of price setting for this interpretation.

\(^{22}\)Equation (13) becomes \(\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa \hat{Y}_t + u_t\).

\(^{23}\)The rationale for these values is discussed in Eggertsson and Egiev (2020).

\(^{24}\)In that paper, a cost push is traced back to the effect of the debt-deleveraging shock on the labor supply of borrowers and savers.
interprets itself as adhering to a dual mandate, with inflation only being one component and some measure of economic activity the other. These two objectives are typically put on equal footing. Thus, one could argue that an equal weight better captures the behavior of the Fed in practice. Second, recent research suggests that once one incorporates realistic idiosyncratic shocks in firms’ pricing decisions, then the weight on output relative to inflation increases substantially (see for example Burstein and Hellwig 2008 and Nakamura et al. 2018).

Figure 3: Selected rules and optimal commitment (OCP) under baseline calibration with cost push shock. Colored lines are contingency 10 for output ($\hat{Y}$), inflation ($\pi$), the nominal interest rate ($i$), and nominal GDP ($\hat{N}$). The vertical axes report deviations from steady state, in percentage points (annualized figures). The vertical axis for the nominal interest rate reports annualized percentage points.

Figure 3 contrasts the optimal policy commitment (OCP) to the Taylor rule (TTR) under this alternative calibration strategy, showing the contingency when the natural rate of interest reverts back to steady state ten quarters after the shock hits (other contingencies look qualitatively similar). Assuming the Federal Reserve follows a Taylor rule implies, by construction, a fall in inflation of 0.5 percent and a drop in output of 7.5 percent. The optimal commitment, in contrast, results in only a two-percent drop in output and, instead of falling, inflation overshoots its target substantially throughout the duration of the shock. The way the Federal Reserve accomplishes this is by committing to keeping the nominal interest rate at the ELB by about six additional quarters after the natural rate normalizes, a similar commitment as in the original EW2003 example. Interestingly, this commitment now implies that during the period of the shock inflation overshoots its target by about 3 percentage points and then mildly undershoots.
it once the shock subsides. This is in sharp contrast to the EW2003 calibration, where there is no such overshooting of the inflation target when the shock hits. The reason for the different result is that the central bank is now putting output deviation and inflation on equal footing in its objective. Accordingly, it is more willing to tolerate higher inflation in order to achieve better output stabilization.

Figure 3 also shows the outcome if the central bank follows a price level target of the form suggested by EW2003 (PLT) and a nominal GDP target (NGDPT). As the Figure suggests, these commitment do not substantially improve the outcome relative to the Taylor rule. The reason is that the small fall in the price level in this numerical experiment leads to a trivial additional commitment to low future rates once the shock has subsided. The key mechanism for why the price level target succeeds in replicating the optimal commitment in the EW2003 calibration lies in the shock generating enough deflation. This means that once the shock is over the central bank does not raise the interest rate for a considerable period of time, or until the price level recovers to the target. If there is very little fall in the price level, this commitment loses most of its power. As it can be seen in the third panel, the nominal interest rate is almost the same for the Taylor rule relative to the price level target or the nominal GDP target. Note that this problem is attenuated by the fact that once a central bank commits to either of these two policies, the equilibrium drop in the price level is even smaller than under the Taylor rule, thus implying an even smaller commitment to future expansion once the shock reverts to steady state.

A stark assumption in the calibration of the shock process is that the Federal Reserve followed a Taylor rule during the crisis. This interpretation implies that the forward guidance done by the Fed during the crisis did not substantially commit to keeping future rates lower as inflation started to recover. By contrast, if we calibrate the model taking the other extreme – that policy was conducted under optimal commitment – then the implied fall in the price level would have been larger under alternative policy regimes, such as the Taylor rule, and thus the price and nominal GDP targeting would have worked better. We prefer our specification, however, as it more clearly highlights possible pitfalls of these targeting strategies. They critically rely upon a sizeable fall in inflation in order to generate any meaningful commitment to low future rates. In the presence of cost push shocks, or very low values for \( \kappa \), there is no reason to expect such a fall in the price level from the perspective of the model.

4.4 Cumulative Nominal GDP Targeting and Symmetric Dual-Objective Targeting

We now consider simple alternatives to the price level and nominal GDP targeting rules that better replicate the optimal commitment policy.

The key problem with both rules was that if there is little fall in the price level, neither rule implies much accommodation once the shock giving rise to the ELB normalizes. The optimal commitment, instead, mandates that if there is fall in either output or the price level for the duration of the ELB, then there should be accommodation (or make-up accommodation) once the shock is over, as seen in the analytic derivation of the fully optimal targeting rule. In that derivation, the time varying target is defined in terms of weighted average of the price level and output. Moreover, if the target is missed, the fully optimal rule suggests that the future target should be increased, thus generating expectations of future accommodation.

Motivated by this observation, let us first consider the following simple targeting rule.\(^{25}\) Define the

\(^{25}\)This rule is closely related to Reifschneider and Williams (2000), see Section 4.5.
Figure 4: Selected rules and optimal commitment (OCP) under baseline calibration with cost push shock. Colored lines are contingency for output ($\hat{Y}$), inflation ($\pi$), the nominal interest rate ($i$), and nominal GDP ($\hat{N}$). The vertical axes report deviations from steady state, in percentage points (annualized figures). The vertical axis for the interest rate reports annualized percentage points.
cumulative deviations of nominal GDP from its trend as $\hat{\Gamma}_t$:

$$\hat{\Gamma}_t = \hat{p}_t + \hat{y}_t + \hat{\Gamma}_{t-1}$$  \hspace{1cm} (27)

This variable measures how much nominal GDP deviates from its target. Relative to previous nominal GDP targeting proposals, such as those previously cited, this variable keeps track not only of the size of deviations of current nominal GDP from its trend, but it also accounts for past misses. The proposed targeting rule is then to set the nominal interest rate so that the cumulative nominal GDP is on trend, i.e. $\hat{\Gamma}_t = 0$, whenever possible. Otherwise, the central bank should set the nominal rate at its effective lower bound, with the threshold for liftoff being that the cumulative nominal GDP target is reached again. Critically, if this rule is credible, the public understands that if the Federal Reserve misses its nominal GDP target, it is then committed to keeping the interest rate at zero until it has compensated for having missed its target. Thus, if it were to miss on trend nominal GDP by 5 percent, it is committed to overshoot trend nominal GDP by 5 percent going forward.

Figure 4 shows how this History-Dependent Nominal GDP Target (HD-NGDPT) does in the numerical experiment. It does substantially better than PLT or NGDPT reported in Figure 3 (we provide a more detailed assessment of this comparison in Table 2). The key to the success of this rule is that it prescribes a substantial easing once the ELB is no longer binding on account of the exogenous shock, much as prescribed by the optimal commitment.

As an alternative to keeping track of how nominal GDP misses its trend, we also consider the following: let us define an index ($\hat{D}_t$) that measures how inflation and real output deviates from trend:

$$\hat{D}_t = 4\hat{\pi}_t + \hat{y}_t + \hat{D}_{t-1}$$  \hspace{1cm} (28)

A targeting rule we coin Symmetric Dual-Objective Targeting Rule (SDTR) sets interest rates so that the Dual Mandate Index $\hat{D}_t$ is set to zero if possible but keeps the nominal interest rate at the ELB otherwise. Critically then, if a central bank following this reaction function misses its target, it will automatically commit to a future accommodation.

Figure 4 shows the performance of this rule and illustrates that it does even better than the cumulative nominal GDP target. Again, the key behind this success is that it implies a considerable easing once the ELB is no longer binding, much beyond price or nominal GDP targets. PLT and NGDPT imply make-up behavior for past misses of the inflation target only. The two new rules show this feature as well, but in addition they do the same for output: if there is a recession today, the central bank then commits to a boom in the future. This type of commitment is particularly important when the price level moves by small amounts.

Overall, the fact that SDTR does better than HD-NGDPT is not robust once we consider richer model such as the FRBNY DSGE. Before getting there, however, we compare these rule to other well-known policy rules and offer a more detailed assessment of their performance.
<table>
<thead>
<tr>
<th>Name</th>
<th>Acronym</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor Rule</td>
<td>TTR</td>
<td>$i_{TTR}^{TTR} = \max {0, \bar{r} + \bar{\pi} + \phi_{\pi} \bar{\pi}_t + \phi_y \hat{Y}_t}$</td>
</tr>
<tr>
<td>with lag</td>
<td>TR(TTR-1)</td>
<td>$i_{TTR}^{TTR-1} = \max {0, \bar{r} + \bar{\pi} + \phi_{\pi} \bar{\pi}<em>{t-1} + \phi_y \hat{Y}</em>{t-1}}$</td>
</tr>
<tr>
<td>with lead</td>
<td>TR(TTR+1)</td>
<td>$i_{TTR}^{TTR+1} = \max {0, \bar{r} + \bar{\pi} + \phi_{\pi} (\bar{\pi}_t + \phi_y \hat{Y}_t)}$</td>
</tr>
<tr>
<td>Interest Smoothing</td>
<td>TTRS</td>
<td>$i_{TTRS}^{TTR} = \max {0, \phi_{\pi} \pi_t + \phi_y \hat{Y}_t}$</td>
</tr>
<tr>
<td>Price Level Rule</td>
<td>TRP</td>
<td>$i_{TRP}^{TTR} = \max {0, \bar{r} + \pi + \phi_{\pi} \hat{\pi}_t + \phi_y \hat{Y}_t}$</td>
</tr>
<tr>
<td>Nominal GDP Target</td>
<td>NGDPT</td>
<td>$i_{NGDPT}^{TTR} [\hat{P}_t + \hat{Y}_t] = 0$</td>
</tr>
<tr>
<td>Augmented Taylor Rule</td>
<td>ATR</td>
<td>$i_{ATR}^{TTR} = \max {0, i_{TR}^{TTR} - \alpha Z_t}$, $i_{TR}^{TTR} = \bar{r} + \bar{\pi} + \phi_{\pi} \bar{\pi}_t + \phi_y \hat{Y}_t$</td>
</tr>
<tr>
<td>Superinertial Taylor Rule</td>
<td>SUP</td>
<td>$i_{SUP}^{TTR} = \max {0, (1 - \phi_{SUP}) (\bar{r} + \pi) + \phi_{SUP} \bar{\pi}_t + \phi_y \hat{Y}_t}$</td>
</tr>
<tr>
<td>Average Inflation Targeting</td>
<td>AIT</td>
<td>$i_{AIT}^{TTR} = \max {0, \bar{r} + \bar{\pi} + \phi_{\pi} \bar{\pi}_t + \phi_y \hat{Y}_t}$</td>
</tr>
<tr>
<td>Price level target</td>
<td>PLT</td>
<td>$i_{PLT}^{TTR} [\hat{Y}_t + \xi \hat{\pi}_t] = 0$</td>
</tr>
<tr>
<td>History-Dependent NGDP Target</td>
<td>HD-NGDPT</td>
<td>$\hat{\pi}_t^{HD-NGDPT} = 0$, $\hat{Y}_t = \hat{\pi}_t + \hat{Y}<em>t + \hat{\pi}</em>{t-1}$</td>
</tr>
<tr>
<td>Symmetric Dual-Objective</td>
<td>SDTR</td>
<td>$D_{SDTR}^{TTR} = 0$, $\hat{D}_t = 4 \hat{\pi}_t + \hat{Y}<em>t + \hat{D}</em>{t-1}$</td>
</tr>
</tbody>
</table>

Table 1: Policy rules, names, and acronyms. $\bar{\pi}_{NN}$ is average inflation over last $NN$ quarters. $\hat{\pi}$ is deviation of price level from its (detrended) steady state value. $\hat{\pi}_t$ and $\hat{\pi}_{t-1}$ are defined in Equation (27) and (28).
4.5 Comparison to Other Policy Rules

In this Section we compare the two policy rules we have suggested to several reaction functions that have been proposed in the literature (Table 1). We summarize their performance in Table 2, using the welfare criterion specified in (26), as well as other measures. The first column shows the welfare loss implied by these different rules, with the optimal monetary policy commitment normalized to 1.

Of the rules considered, the best performing ones are the two we have introduced in Section 4.4, together with the Augmented Taylor Rule (ATR) proposed by Reifsneider and Williams (2000) and the Superinertial Taylor Rule (SUP) described in Rotemberg and Woodford (1999). Figure 5 depicts the dynamic response of the simple two-equation model under these different reaction functions, alongside two benchmarks: the optimal commitment policy (OCP) and a truncated Taylor Rule (TTR). We discuss each one in turn.

The Augmented Taylor Rule is closely related to our two proposals. According to this rule, a cumulative index \( Z_t \) keeps track of how much the actual interest rate “misses” the interest rate suggested by the standard Taylor rule due to the ELB. This reaction function then prescribes the interest rate to respond to these cumulative misses at a rate \( \alpha \). In turn, this implies that if the ELB is binding, future rates will be lower than predicted by a standard Taylor rule to make up for previous deviations from target. This is a similar make-up feature to the one characterizing the fully optimal commitment rule in EW2003, as well as our two proposed rules HD-NGDPT and SDTR.

The Superinertial rule of Rotemberg and Woodford (1999) generates a similar commitment to low rates for a prolonged period of time, but for different reasons. As shown in Table 1, the super-inertial rule has a lagged interest-rate term appearing in an otherwise standard Taylor rule. The coefficient on this lagged interest rate, however, is greater than one – hence the name super-inertial. This implies not only that interest rates drop very slowly in reaction to the shock, as shown in Figure 5, but also that they increase equally sluggishly once the shock reverts. This generates exactly the type of commitment needed. Interestingly, even though this rule does not prescribe an immediate drop in the nominal interest rate in reaction to the shock – as the optimal commitment would mandate – it still outperforms most of the other policy functions under consideration.

In line with the finding of Reifsneider and Wilcox (2019), we can also document how a Taylor-type rule reacting to average inflation (like AIT) is suboptimal at best, with a welfare loss comparable to a price-level or nominal-GDP target. Again, this is mostly due to the limited drop in inflation that prevails in this parametrization of the model.

In the next sections we assess the robustness of this conclusion to different assumptions in terms of size of the inflation response, its source, choice of policy parameters, as well as how our results transpose to a more sophisticated macroeconomic model.

4.6 Robustness Checks

In the previous Sections we explored the performance of our novel policy rules, and we found that they improve upon several alternatives documented in the literature, such as PLT and NDGPT. We did so in an environment in which inflation on impact was slightly below steady state under a Taylor rule (-0.5%). This reflects what the US economy has experienced during the Great Recession episode of 2007-08.

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\(^{26}\)The rule is closely related to Reifsneider and Williams (2000), see Section 4.5, and rules that feature a lagged interest rate, as in Taylor and Williams (2010) or recently discussed in Kiley and Roberts (2017).

\(^{27}\)The inflation gap is multiplied by 4 to make sure that there is equal weight on annualized inflation and output.
Figure 5: Dynamic response to a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% constant under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Table 2: Some metrics for selected interest rate rules the simple two-equation NK model in the presence of a natural rate shock and a correlated cost push shock. All rows except the first show values normalized with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function \( (26) \) for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Effective Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output and inflation to a natural interest rate shock and a correlated cost push shock such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). The model is calibrated with the standard EW2003 parameter values reported in footnote 19. The list of acronyms is detailed in Table 1. TTR+1 coincides with TTR in the simple two-equation NK model.

We repeat the experiment with different parametrizations of the natural rate and cost push shock to have lower (0%) or higher deflation (-2%) on impact under TTR. In both scenarios our result holds – namely that HD-NGDPT and SDTR fare relatively well when compared to PLT and NGDPT. In particular, the welfare loss under HD-NGDPT (SDTR) is approximately 1.6 (1.2) times as large as in the optimal commitment policy. On the contrary, the welfare loss in both PLT and NGDPT is at least three times as large as in OCP. In Sections A.6.2 and A.6.3 in the Appendix, we report the detailed performance assessment of our candidate rules, as well as plots for the impulse responses.

In the experiments so far the inflation response has been quite weak due to the presence of a cost push shock perfectly correlated with the drop in the natural interest rate. An alternative theory for why inflation was so stable in the US Great Recession episode points to price rigidity. We explore this avenue by repeating the exercise in a standard two-equation model with no cost push shock, but with a degree of price rigidity such that, under TTR, inflation again drops to -0.5%. Again, we refer the reader to the Appendix (Section A.6.4) for welfare analysis and dynamic responses. Our two rules remain very good compared to the optimal policy commitment. On the other hand, rules that were comparable to ours, ATR and SUP, now tend to fare significantly worse than OCP.

The last robustness check we perform involves the policy parameters: we choose them to maximize welfare according to (26).\(^\text{28}\) Optimal parameters are reported in Table A.2. In this setting, only ATR and SUP – namely those rules that display welfare losses comparable to ours – improve slightly in terms of welfare. All other optimized policies are virtually unchanged.

\(^{28}\)Targeting rules are not re-parametrized.
4.7 Medium Scale DSGE Model

The results discussed so far were derived under a simple two-equation NK model. In this Section we show that the findings generalize to a medium scale DSGE model and they are not an artifact of the simple structure of the baseline exercise. This Section also serves to illustrate how the toolkit can easily handle medium-scale models.

Rules that imply substantial make-up behavior or feature inertia in the interest rate path show the best outcome. Our two proposed rules, HD-NGDPT and SDTR, outperform all other policies by implying substantial stimulus compared to a simple Taylor rule. As before, a price level target and nominal GDP target do not perform as well in an environment with a relatively stable price level.

For this exercise, we implement with our toolkit the FRBNY DSGE model as outlined in Del Negro, Giannoni and Patterson (2013). We calibrate the model to observed output and inflation declines in the US Great Recession. In all our figures, we plot the data against our simulations. Our results therefore could also be read as counterfactual outcomes had the Fed adhered to different policy rules. Section A.5 in the Appendix contains details on the model and the calibration.

We start by checking the performance of a price level target and a nominal GDP target relative to our newly proposed rules, as in Sections 4.3 and 4.4. Like before, we are interested in an economy that exhibits a relatively modest drop of inflation while the economy is at the ELB.

Figure 6 compares a simple Taylor rule and the four targeting rules.\footnote{In this setting, the optimal commitment policy is no longer our benchmark: in a medium scale model like the FRBNY DSGE featuring many state variables, optimal commitment is cumbersome to derive.} We see that PLT and NGDPT do not improve significantly upon the standard Taylor rule. The reason is the usual one: the small drop in the price level does not generate enough commitment to low future rates. The Taylor rule implies 33 quarters at the ELB. Relative to this, PLT and NGDPT only command four additional quarters at the lower bound. This is in stark contrast to our newly proposed rules: HD-NGDPT (SDTR) implies 15 (14) additional quarters at the ELB relative to the Taylor rule, postponing the liftoff from zero well into the year 2019. This aggressive policy improves outcomes considerably: while output drops to $-7.4\%$ under the Taylor rule, and still to $-6.3\%$ ($-6.0\%$) under PLT (NGDPT), the maximum drop in output under HD-NGDPT (SDTR) is $-3.6\%$ ($-3.1\%$). In addition, we should note that output hits its trough early, and recovers thereafter. This improvement is achieved at the expense of a modest overshoot of inflation relative to the other rules.

Table 3 and Figures 7 and A.18 show the results for the full set of reaction functions.\footnote{Appendix A.6.7 shows impulse response functions and additional variables for all rules.} We calibrate the model so that under the FRBNY rule (Del Negro, Giannoni and Patterson (2013), Equation A.7) the drop in output and inflation matches the data.

Comparing the welfare loss across rules, we see that HD-NGDPT and SDTR outperform all others. Our proposals imply a welfare loss that is only a third of the one under the FRBNY rule, since they imply substantial additional stimulus: the expected duration at the ELB (Column 2 in Table Table 3), is much longer than any alternative proposal. Of the remaining rules, ATR and SUP again show good performance and come closest to our two proposed rules.
Figure 6: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under different policy rules. Colored lines show paths for output ($\hat{Y}_t$), inflation ($\pi$), the nominal interest rate ($i$), and nominal GDP ($\hat{N}$). Dotted red line is data. The two-state Markov shocks switch to low state in Q4-07 and revert to the absorbing state after 32 quarters (32nd contingency). The vertical axes for $\hat{Y}_t$ and $\hat{N}$ report deviations from detrended steady state, in percentage points (annualized figures). The vertical axes for $\pi$ and $i$ report annualized percentage points. The horizontal axis shows quarter and calendar year. See Section A.3 for details on data and Section A.5.2 for calibration. The list of acronyms is detailed in Table 1.
Figure 7: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under baseline policy rules. Colored lines show paths for output ($\hat{Y}_t$), inflation ($\pi$), the nominal interest rate ($i$), and nominal GDP ($\hat{N}$). Dotted red line is data. The two-state Markov shocks switch to low state in Q4-07 and revert to the absorbing state after 32 quarters (32nd contingency). The vertical axes for $\hat{Y}_t$ and $\hat{N}$ report deviations from detrended steady state, in percentage points (annualized figures). The vertical axes for $\pi$ and the $i$ report annualized percentage points. The horizontal axis shows quarter and calendar year. See Section A.3 for details on data and Section A.5.2 for calibration. The list of acronyms is detailed in Table 1. FRBNY rule refers to Equation (A.7).
### Table 3: Some metrics for interest rate rules in the FRBNY model in the presence of a preference shock and correlated cost push shock. All rows except the first show values normalized with respect to the modified Taylor rule in Del Negro, Giannoni and Patterson (2013) (FRBNY Rule, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights and an inflation target, see Equation (A.8); Column (2) displays the unconditional expected duration of the Effective Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a preference shock and a correlated cost push shock such that output falls by 8.5% and inflation by 0.5% under FRBNY Rule. See Section A.5.2 for details on calibration. The list of acronyms is detailed in Table 1.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Welfare Loss</th>
<th>$E_0[\tau + k_T - \bar{T}]$</th>
<th>Volatility $\bar{Y}$</th>
<th>Volatility $\pi$</th>
<th>Impact $\bar{Y}$</th>
<th>Impact $\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FRBNY Rule</td>
<td>0.001</td>
<td>0.855</td>
<td>0.013</td>
<td>1.350 $10^{-5}$</td>
<td>7.385 $10^{-5}$</td>
<td>-2.525</td>
</tr>
<tr>
<td>TTR</td>
<td>0.772</td>
<td>5.552</td>
<td>0.765</td>
<td>1.214</td>
<td>2.182</td>
<td>0.866</td>
</tr>
<tr>
<td>HD-NGDPT</td>
<td>0.340</td>
<td>15.132</td>
<td>0.319</td>
<td>1.680</td>
<td>5.081</td>
<td>0.603</td>
</tr>
<tr>
<td>SDTR</td>
<td>0.330</td>
<td>15.689</td>
<td>0.303</td>
<td>1.987</td>
<td>5.130</td>
<td>0.599</td>
</tr>
<tr>
<td>ATR</td>
<td>0.457</td>
<td>7.611</td>
<td>0.441</td>
<td>1.477</td>
<td>2.662</td>
<td>0.778</td>
</tr>
<tr>
<td>SUP</td>
<td>0.435</td>
<td>5.194</td>
<td>0.417</td>
<td>1.535</td>
<td>2.788</td>
<td>0.675</td>
</tr>
<tr>
<td><strong>Panel A:</strong> baseline rules</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PLT</td>
<td>0.600</td>
<td>10.321</td>
<td>0.590</td>
<td>1.222</td>
<td>3.137</td>
<td>0.773</td>
</tr>
<tr>
<td>NGDPT</td>
<td>0.581</td>
<td>9.936</td>
<td>0.571</td>
<td>1.205</td>
<td>3.065</td>
<td>0.772</td>
</tr>
<tr>
<td>TTRP</td>
<td>0.852</td>
<td>1.320</td>
<td>0.850</td>
<td>0.989</td>
<td>1.051</td>
<td>0.955</td>
</tr>
<tr>
<td>TTRS-1</td>
<td>0.603</td>
<td>2.470</td>
<td>0.592</td>
<td>1.312</td>
<td>1.724</td>
<td>0.849</td>
</tr>
<tr>
<td>TTR-1</td>
<td>0.712</td>
<td>4.774</td>
<td>0.703</td>
<td>1.246</td>
<td>2.067</td>
<td>0.860</td>
</tr>
<tr>
<td>AIT</td>
<td>0.707</td>
<td>5.109</td>
<td>0.700</td>
<td>1.108</td>
<td>2.060</td>
<td>0.856</td>
</tr>
<tr>
<td>FRBNY Rule</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>TTR+1</td>
<td>0.852</td>
<td>6.184</td>
<td>0.847</td>
<td>1.175</td>
<td>2.252</td>
<td>0.889</td>
</tr>
<tr>
<td><strong>Panel B:</strong> additional rules</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.8 Other Applications

In this Section we highlight some features and characteristics of the toolkit by the use of simple examples, and highlight the value added of our framework relative to other toolboxes, such as *Occbin*, relying on deterministic algorithms. We provide two short examples in the context of a standard two-equation NK model with a negative shock to the natural rate, to explain: (1) the stochastic nature of our solution method; (2) how one can use the toolkit to study different sources of disturbances that can generate the same observable effect, i.e. an increase in the expected duration of the ELB.

In the first example, we show how a (optimal) time-contingent, or fixed-length, forward guidance policy (FLFG)$^{31}$ closely tracks the optimal commitment policy under a deterministic AR(1) structure both in terms of impulse responses and welfare loss, as opposed to the two-state Markov framework. This is to highlight that an analyst who only considers a deterministic process for the underlying shock might be tempted to propose a fixed-length policy commitment for the nominal rate while the stochastic structure illustrates that this policy is severely suboptimal if there is uncertainty about the duration of the shock.

The second example compares two types of forward guidance, *Odyssean* and *Delphic*. $^{32}$ In the former case, the monetary authority announces a flat extra stimulus, i.e. a constant increase of the vector $k$ compared to the original commitment; the latter is a case in which the expected duration of regimes 1 and 2 increases by the same amount as in the previous case, but for exogenous reasons. This can be due to the central bank signaling the exogenous shock to the natural rate to be more persistent than...

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$^{31}$The policy is to hold nominal rates at zero until a specific date, and extend this period if needed. This resembles the announcement by the Bank of Canada in April 2009.

$^{32}$The terms *Odyssean* and *Delphic* are motivated in Campbell et al. (2012).
previously anticipated by the other agents in the economy. Our toolkit is a relatively straightforward way of illustrating the effect of these two policies due to the stochastic nature of the shock.

4.8.1 The fixed-length forward guidance experiment

Consider a policy that consists in the monetary authority announcing that, in response to a negative natural rate shock, it will hold the nominal rates at zero until a fixed calendar date or for a fixed amount of time \( \nu \), after which the policy will return to a discretionary regime unless otherwise needed. This corresponds to the following policy for nominal rates.

\[
i_t = \begin{cases} 
0 & \text{for } t \leq \nu \\
\tau + \phi \pi_t & \text{otherwise}
\end{cases}
\]  

(29)

This experiment makes use of a feature of the toolkit: we can choose the \( k \) to alter the expected duration of the ELB, without changing size or transition probability of the exogenous shock.

We consider two possibilities for the process of \( r^\pi_t \), namely an AR(1) structure\(^{33}\) and the two-state Markov structure.\(^{34}\) For each exogenous process we compute the optimal value of \( \nu \) for the policy (29), based on a standard quadratic welfare evaluation.\(^{35}\) We finally compare the effects of the standard Taylor rule, the rule (29) with optimal \( \nu \), as well as OCP. Figure 8 reports the impulse response function for inflation, output, and nominal rate. The left panels corresponds to the two-state Markov process, while the right panels show the results under the AR(1) process.

**AR(1) process** Under the deterministic AR(1) process, the IRF of output and inflation, under the optimal FLFG closely tracks the ones from OCP. The welfare loss is also very close to the one in OCP (1.346 in relative terms). Notably, optimal FLFG is much closer to OCP than to the Taylor rule, both in terms of impulse response and welfare loss.

**Two state Markov** Under the two-state Markov process, the previous result is not valid anymore. The welfare loss implied by the optimal FLFG is much larger than compared to the one in OCP. This result highlights the role of the stochastic structure, i.e. once the shock moves away from a deterministic structure, the optimal FLFG fares much worse than OCP.

4.8.2 Odyssean and Delphic forward guidance

In this example we consider two experiments that increase the expected duration at the effective lower bound by one period. In the first experiment, the Odyssean forward guidance, we impose the following

\(^{33}\)The gap between steady state natural rate and actual natural rate evolves with persistence \(|\rho| < 1\), i.e. \( r^\pi_t - \tau = \rho (r^\pi_{t-1} - \tau) \), after an initial drop, i.e. \( r^\pi_1 < 0 \).

\(^{34}\)The shock processes are calibrated to obtain the same effect, \(-7.5\%\), on output on impact under the standard Taylor rule, namely for \( \nu = 0 \).

\(^{35}\)In this case, as opposed to the previous Sections, we use the welfare loss function as derived in Gali (2015) and Woodford (2003).
Figure 8: Optimal Fixed-Length Forward Guidance (FLFG, red solid line) under alternative stochastic processes for the natural interest rate: two-state Markov (Panel a, left) or an AR(1) process (Panel b, right). The solid lines represent impulse responses for output ($\hat{Y}$), inflation ($\pi$) and the nominal interest rate ($i$). The size of the Markov shock and the persistence of the AR(1) process are calibrated to achieve a drop in output in the initial period of -7.5% under discretion (blue dashed line). The purple solid line reports the optimal policy under commitment (Ramsey plan). The vertical axes for inflation and output reports deviations from steady state, in percentage points (annualized figures). The vertical axis for the interest rate reports annualized percentage points. Welfare losses with respect to optimal commitment are reported for FLFG ($W_{FLFG}$) and ODP ($W_{ODP}$).
This is an expansionary experiment compared to the standard Taylor rule (TTR). In the second experiment, the *Delphic* forward guidance, we adjust $\mu$, i.e. the probability of the Markov disturbance, to obtain the same increase of one to the expected duration of the ELB. In order to obtain a longer duration $\mu$ has to increase. This is a more contractionary shock compared to before. The two experiments are meant to separately study the effects of policy (Odyssean) and the effect of news or beliefs about fundamentals of the economy (Delphic).

Results are shown in Figure 9. The calibration is the same as used in Section 4.3, i.e. equal welfare weights and a drop of output by 7.5% and inflation by 0.5%.

**Odyssean Forward Guidance** In this experiment, we again “exogenously” change the duration of the ELB with a (credible) announcement by the central bank to keep nominal interest rates lower for longer than as implied by the Taylor rule. Private sector agents fully internalize this information and optimally react to the new interest rate path (which they take as perfectly credible), correctly taking into account the uncertainty connected to the length of the shock spell that is unchanged to before.

**Delphic Forward Guidance** This experiment increases the expected duration of the ELB episode by changing the transition probability of the Markov process. We can think of this as the realization of bad news (probably through revelation by an agent with superior information, e.g. the central bank) about fundamentals of the economy (expected length of shock). Note that this experiment also highlights the crucial role of expectations in the model: even if we would not change the objective transition probability, if agents subjectively believe that the transition probability has changed, and act accordingly, this is enough to induce a movement in aggregate variables.
Figure 9: Odyssean vs. Delphic forward guidance (FG) experiment in simple two-equation NK model. Odyssean FG features commitment of central bank to one additional periods at ELB relative to Taylor rule (TTR). Delphic FG features lower transition probability of shock, $1 - \mu$. Both experiments have expected duration at ELB of 11 quarters, TTR has expected duration at ELB of 10 quarters. Left panel shows contingency 10, right panel impulse response functions. The colored lines represent path for output ($x$), inflation ($\pi$) and the nominal interest rate ($i$). Calibration is the same as in Section 4.3, achieving a drop in output in the initial period of -7.5% and a fall in inflation to $-0.5\%$ under discretion in response to a natural rate shock and a perfectly correlated cost push shock (TTR, blue solid line). Transition probability under Delphic FG is $\mu = 0.09091$. The vertical axes for inflation and output reports deviations from steady state, in percentage points (annualized figures). The vertical axis for the interest rate reports annualized percentage points.

5 Conclusions

We provide a toolkit to solve DSGE models that involve occasionally binding constraints. The solution method generalizes that of Eggertsson and Woodford (2003) and exploits the properties of a two-state Markov process for the exogenous disturbances. The toolkit performs well even in the presence of a large number of state variables and features a fully stochastic structure.

We use the toolkit to study the performance of policy rules in economies that experience the ELB. Our two newly proposed rules, a History-Dependent Nominal GDP Target and a Symmetric Dual-Objective Target, consistently outperform most rules documented in the literature, especially reaction function based on price-level or nominal-GDP targeting. We show that the latter two lack sufficient stimulus in recessions characterized by small drop in the price level, as experienced in the recent US Great Recession episode.
References


A Appendix

A.1 Proof of Lemma 1

\textit{Proof.} The system can be written as:
\[
\begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}
\begin{bmatrix}
E_t Z_t + 1 \\
P_t
\end{bmatrix}
= \begin{bmatrix}
B_1 & B_2 \\
B_3 & B_4
\end{bmatrix}
\begin{bmatrix}
Z_t \\
P_{t-1}
\end{bmatrix}
\]
\[
\begin{bmatrix}
A_1 & A_2 & -B_1 & -B_2 \\
A_3 & A_4 & -B_3 & -B_4
\end{bmatrix}
\begin{bmatrix}
E_t Z_{t+1} \\
P_t \\
Z_t \\
P_{t-1}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
\[
\begin{bmatrix}
A_2 & -B_1 & -B_2 & A_1 \\
A_4 & -B_3 & -B_4 & A_3
\end{bmatrix}
\begin{bmatrix}
P_t \\
Z_t \\
P_{t-1}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] \quad (A.1)

Let the row reduced echelon form of \[
\begin{bmatrix}
A_2 & -B_1 & -B_2 & A_1 \\
A_4 & -B_3 & -B_4 & A_3
\end{bmatrix}
\] be \[
\begin{bmatrix}
I & 0 & -C_1 & -C_2 \\
0 & I & -C_3 & -C_4
\end{bmatrix}
\]
The system will then be:
\[
\begin{bmatrix}
I & 0 & -C_1 & -C_2 \\
0 & I & -C_3 & -C_4
\end{bmatrix}
\begin{bmatrix}
P_t \\
Z_t \\
P_{t-1}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
\[
\begin{bmatrix}
P_t \\
Z_t \\
P_{t-1}
\end{bmatrix}
= \begin{bmatrix}
C_1 & C_2 \\
C_3 & C_4 \\
E_t Z_{t+1}
\end{bmatrix}
\] \quad (A.2)

At time $t + 1$, recall that $j \equiv k \tau - (t - \tau)$, we know that:
\[
P_{t+1} = G^{2,j-1} P_t
\]
\[
Z_{t+1} = D^{2,j-1} P_t
\]
It follows that $E_t Z_{t+1} = E_t D^{2j-1} P_t = D^{2j-1} P_t$. By substituting this result in (A.2) we can solve the system:

$$
\begin{bmatrix}
P_t \\
Z_t
\end{bmatrix} =
\begin{bmatrix}
C_1 & C_2 \\
C_3 & C_4
\end{bmatrix}
\begin{bmatrix}
P_{t-1} \\
E_t Z_{t+1}
\end{bmatrix}
= \begin{bmatrix}
P_t \\
Z_t
\end{bmatrix} = \begin{bmatrix}
C_1 P_{t-1} + C_2 D^{2j-1} P_t \\
C_3 P_{t-1} + C_4 D^{2j-1} P_t
\end{bmatrix}
$$

(A.2)

$$
\begin{bmatrix}
P_t \\
Z_j
\end{bmatrix} = \begin{bmatrix}
1 & \sigma \\
0 & \beta
\end{bmatrix}
\begin{bmatrix}
P_{t-1} \\
E_t Z_{t+1}
\end{bmatrix}
= \begin{bmatrix}
P_t \\
Z_j
\end{bmatrix} = \begin{bmatrix}
(I - C_2 D^{2j-1})^{-1} C_1 P_{t-1} \\
(C_3 + C_4 D^{2j-1} (I - C_2 D^3)^{-1} C_1) P_{t-1}
\end{bmatrix}
$$

(A.3)

$$
\begin{bmatrix}
P_t \\
Z_j
\end{bmatrix} = \begin{bmatrix}
G^{2j} P_{t-1} \\
(C_3 + C_4 D^{2j-1} G^{2j}) P_{t-1}
\end{bmatrix}
$$

(A.4)

\[36\]

**A.2 Code Setup**

In this section we explain how to set up codes for the toolkit for the New Keynesian model under OCP as explained in Section 4. The solution algorithm is generated in three functions that have to be run sequentially. The essential inputs of those functions are (1) the matrices that define the model; (2) a parameter structure that is described below; (3) a configuration structure that is described below. There are several optional inputs that we explain at the end of the Section.

**A.2.1 Matrices**

Consider Equations (12)-(18) in the absorbing state, where (16) is strictly binding. The system can be written as $A E_t \times \xi_{t+1} = B \times \xi_t$, where $\xi_t \equiv [Y_t, \pi_t, i_t, \phi_{1t-1}, \phi_{2t-1}, r^n_{t-1}, u_t]$ and

$$
A = \begin{bmatrix}
1 & \sigma & 0 & 0 & 0 & \sigma & 0 \\
0 & \beta & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & \kappa & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix},
B = \begin{bmatrix}
1 & 0 & \sigma & 0 & 0 & 0 & 0 \\
-\kappa & 1 & 0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & -\frac{1}{\beta} & 0 & 0 & 0 \\
0 & 1 & 0 & -\frac{1}{\beta} \sigma & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(A.5)

**Important** Note that the user should order elements in the vector $\xi_t$ such that the jump variables come first, the nominal rate should be between jump variables and predetermined variables, predetermined variables should follow, and finally shock variables should be at the end of the vector.$^{37}$

$^{36}$We suggest to use $AAA$ and $BBB$ in place of $A$ and $B$ in order to avoid coding conflicts with the inputs in other scripts.

$^{37}$Note that if the system has constants, one can implement them in 2 ways. First, the user could define a predetermined variables along with its own initial value, that never change value across time. A second way is to increase the shock vector to include a shock that does not change value across regimes.
A.2.2 Parameters structure

The parameter structure (we name it \textit{param}) needs to contain the following elements.

1. A scalar element $\mu$, equal to the Markov probability described in the text.
2. Two vector elements $r_h$ and $r_l$. Equal to the values of the two-state Markov process in the absorbing and crisis states. The order needs to match the order in the vector $\xi_t$.
3. A scalar element $NS$, that is equal to the number of predetermined variables in $\xi_t$, excluding the exogenous shocks.

A.2.3 Configuration structure

The configuration structure (we name it \textit{config}) needs to contain the following elements.

1. A scalar element $\tau_{\text{max}}$, equal to the time at which the Markov shock reverts to its absorbing state with probability 1, conditional on being in the crisis state at $\tau_{\text{max}} - 1$.
2. A scalar element $\text{maxlength}2$, equal to the maximum length of regime 2. This is a shortcut to save computing time, but is essentially non material. The toolkit will warn the user if regime 2 requires a higher value for $\text{maxlength}2$.
3. A scalar element $\text{bound}$, equal to lower bound in the inequality constraint, e.g. 0 for the zero lower bound.
4. A scalar element $\text{mono}$. The user shall set $\text{mono}=1$ if the toolkit should be run with the monotonicity assumption on the vector $k$.

A.2.4 The general setup

The user should construct the elements described above and set the code as follows.

```matlab
[D_3,G_3,D_3a] = regime3(AAA,BBB,param);
[D_2,G_2] = regime2(AAA,BBB,D_3a,param,config);
[D_1,G_1, ResM, max_k,k,T_tilde] = ... 
   regime1(AAA,BBB,D_3a,D_3,D_2,G_3,G_2,param,config);
impulseresponse
```

A.2.5 The function \texttt{regime3.m}

The function \texttt{regime3.m} takes as inputs the matrices $AAA$ and $BBB$, as well as the parameters structure $\text{param}$.

```matlab
[D_3,G_3,D_3a] = regime3(AAA,BBB,param);
```

This function provides transition matrices that are then used in the other functions.
A.2.6 The function `regime2.m`

The function `regime2.m` takes as inputs the matrices \( AAA \) and \( BBB \), the parameters structure `param`, the configuration structure `config`, as well as one output \((D_{3a})\) from the function `regime3.m`.

\[
[D_2, G_2] = \text{regime2}(AAA, BBB, D_{3a}, param, config);
\]

This function provides transition matrices that are then used in the function `regime1.m`.

A.2.7 The function `regime1.m`

The function `regime1.m` takes as inputs the matrices \( AAA \) and \( BBB \), the parameters structure `param`, the configuration structure `config`, as well as the output from the functions `regime3` and `regime2.m`.

\[
[D_1, G_1, ResM, \text{max}_k, k, \tilde{T}] = \ldots \\
\text{regime1}(AAA, BBB, D_{3a}, D_2, G_3, G_2, param, config);
\]

This function provides a 3 dimensional matrix \( ResM \), a scalar \( \text{max}_k \), the vector \( k \), the scalar \( \tilde{T} \), and the transition matrices in `regime1.m`.

The dimensions of \( ResM \) are time, variables, and contingencies. The element \( ResM(5, 1, 8) \) contains the value of the variable in position 1 in \( \xi_t \), at time 5, for contingency 8.

The vector \( k \) is a vector that links contingencies and their respective duration of regime 2. \( \text{max}_k \) is the maximum value across \( k \), and the scalar \( \tilde{T} \) is the period at which regime 1 begins.

Optional inputs  There are several optional parameters to `regime1.m`. The user can choose to have any combination of the following:

<table>
<thead>
<tr>
<th>Option</th>
<th>Values</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>verbose</td>
<td>0 (default)</td>
<td>display real-time diagnostics from the search algorithm;</td>
</tr>
<tr>
<td>k_input</td>
<td>vector</td>
<td>impose arbitrary ( k );</td>
</tr>
<tr>
<td>( \tilde{T} )_input</td>
<td>scalar</td>
<td>input a ( \tilde{T} ) value choice;</td>
</tr>
<tr>
<td>R0_search</td>
<td>0</td>
<td>(default)</td>
</tr>
</tbody>
</table>

We find that the first option is the least important in terms of economic results. It does significantly improve upon computing time.

The input of a vector \( k \) shall be taken with some caution. Notice that if \( k \) has too low entries (e.g. there is too little stimulus), the toolkit will force \( k \) to change because regime 3 would feature a violation of the inequality constraint. On the other hand, shall one want to impose a \( k \) that is large enough (for instance in the case of a fixed horizon forward guidance experiment), the user should be aware it is necessary to shut down the search for \( \tilde{T} \) at the same time to fulfill the purpose. For the third optional input, an issue arises when the user inputs a value for \( \tilde{T} \) that is too large. The toolkit will not find a solution because of the algorithm.\(^{38}\) To avoid this, one should shut down the search for \( \tilde{T} \) as well. In this case the toolkit will give a solution that features the value of \( \tilde{T} \) chose by the user (otherwise the default value is 1). Note that this solution may violate the inequality constraint in regime 0, exactly because the algorithm for the

\(^{38}\)See Section 3 for more details.
search of $\tilde{T}$ is shut down. Shutting down the regime 0 algorithm, and setting $T\_tilde\_input$ to $t_{max}$ corresponds to the case of the ELB never binding.

A.2.8 The script impulseresponse.m

This script generates impulse response functions as weighted averages of the evolution of each variable across all contingencies, weighted by the ex-ante probability of the shock reverting to the absorbing state. The script generates a two-dimensional matrix, where each row is a period and column is a variable. Notice that the matrix $ResM$ contains variables in levels, so percentage point variations can be obtained by simply multiplying by 100. Notice that in some cases, as for inflation, the variable is defined on a quarterly basis, so it needs to be multiplied by 4 to yield annualized variations.

A.2.9 The function graphing.m

In addition, we provide some basic graphical function that produce plots as in Section 4. The function needs as inputs the following:

- the matrix $IR$;
- a variables structure, that contains the position of each variable;
- a scalar for the horizon of the plots;
- an ordered vector with the names of the variables one wishes to plot;
- the matrix $ResM$;
- a vector containing the contingencies that one wishes to plot.

The line of code below shows an example.

```
graphing(IR,vars,60,["x","pi","i","i_rule","phi1","ai20"],ResM,1:2:60)
```

This line produces the plots for variables $x$, $\pi$ and so on, for contingencies 1 to 60, as well as the impulse response functions.

A.3 Data

Output shows deviation of real GDP from the linear trend of real GDP estimated on 2000Q1-2007Q2 sample. Inflation shows chain-type price index of personal consumption expenditures, percentage change to previous quarter, annualized. We smooth the inflation data series with a simple moving average. The interest rate is the federal funds rate. All series are retrieved from FRED, Federal Reserve Bank of St. Louis. Data series for the price level, NGDP, cumulated NGDP deviations and Dual Mandate index are constructed from output and inflation series.
A.4 Additional welfare metrics

In the Tables providing performance metrics for the policy rules we calculate a volatility index for selected variables $z_t$ as the following.

$$ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (z_t - \bar{z})^2 $$

(A.6)

A.5 FRBNY DSGE model

The FRBNY DSGE model was developed for policy analysis at the Federal Reserve Bank of New York and builds on several milestone papers in the DSGE literature. The model features nominal wage and price rigidities, variable capital utilization, costs of adjusting investment, habit formation in consumption and credit frictions. In total, the equilibrium conditions of the model include 17 equations.

The equilibrium conditions are taken one for one from Del Negro, Giannoni and Patterson (2013). To implement the model in our toolkit, we make different assumptions on the shock structure. Del Negro, Giannoni and Patterson (2013) include 8 structural shocks, each following an AR(1) process, and 2 i.i.d. monetary policy shocks. We assume that the shocks are perfectly correlated and follow a two-state Markov Process with absorbing state. The applications in this paper feature two structural shocks: a preference shock, $\hat{b}_t$ and a cost push shock, $\tilde{\lambda}_{tT}$. The preference shock scales the overall per period utility and acts as a negative shock to the natural rate of interest in our experiment. The cost push shock enters the Phillips curve and is used to target a specific drop in inflation.

The policy rule proposed in Del Negro, Giannoni and Patterson (2013) is the following:

$$ R_t = \max \left\{ 0; \rho_R R_{t-1} + (1 - \rho_R) \left( \varphi_\pi \sum_{j=0}^{3} \hat{\pi}_{t-j} + \varphi_y \sum_{j=0}^{3} (\hat{y}_{t-j} - \hat{y}_{t-j-1}) + \ln \bar{R} \right) \right\} $$

(A.7)

where $R_t$ is the (gross) nominal interest rate, $\pi_t$ inflation rate, $y_t$ output gap and the remaining are parameters. All hatted variables are in log-deviation from steady state, while steady state variables are denoted by a bar. The policy rule in this model has a standard form: The central bank sets the interest rate according to a function of (lagged) terms of inflation and output as well as the nominal interest rate of the previous period. If this number turns out to be negative, the nominal interest rate is equal to zero, the lower bound. In regimes 1 and 2, the lower bound part of the policy rule will be in effect. In regimes 0 and 3, the endogenous part of the policy rule is part of the equilibrium conditions and has to hold at a candidate solution.

The model includes several predetermined variables introducing inertial dynamics into the model. A consequence of this is that we can no longer impose the monotonicity of $k$: As the shock is on for longer, the state variables, like capital, approach their new ‘steady state’. As they do this, it can turn out to be optimal to keep the interest rate at the lower bound for a shorter period as contingencies get higher, i.e. a hump shaped $k$.

---

39 The FRBNY DSGE Model is explained in detail in Del Negro et al. (2013). We implement a slightly different version of the model which is presented in Del Negro, Giannoni and Patterson (2013). We decide in favor of this version because it features a preference shock.

40 The model includes several lagged terms. Setting up the full model with our toolkit, we count 15 state variables.

41 We will commonly refer to this rule as the “FRBNY rule”.

42 It should also be mentioned that we do not prove analytically that there is a unique $k$ and henceforth a unique equilibrium in the model. We have not encountered any case of multiple equilibria when experimenting with the model.
A.5.1 NYFRB Model description

Please refer to the Appendix of Del Negro, Giannoni and Patterson (2013).

A.5.2 Calibration

To parametrize our model, we choose the posterior means as reported in Del Negro, Giannoni and Patterson (2013) as parameter values.

There are only five exceptions: the transition probability of the two-state Markov shock, \( \mu \), the values of the two shocks in the low state, \( \hat{b}_L \) and \( \hat{\lambda}_{f,L} \), the discount rate \( \beta \) and the steady state inflation rate \( \bar{\pi} \). The first three values are chosen to minimize the quadratic distance to three targets: maximum drop of output of 8.5%, a maximum drop of inflation when at the ELB to a value of 1.5% and an expected duration at the ELB at the point in time of hitting the ELB of 4 quarters. The first two are motivated by observed values during the Great Recession. Expected duration at the ELB of 4 quarters we take from Blue Chip survey of forecaster, see Aspen Publishers 2008-12.

Importantly, these targets have to be matched for a realization of the shock that implies 28 quarters at the ELB, as observed in the data. In our calibration this corresponds to the two-state Markov shock switching to the high state in period 32, i.e. contingency 32.\(^{43}\) The discount rate \( \beta \) is chosen to match a steady state natural rate of real interest of 0%. A real rate of 0 corresponds to the lower bound of December 2019 FOMC long run projections. Finally, the steady state inflation rate is set to 2%. Table A.1 shows the values of the variables we target in the data and the model, respectively, and the implied values for the parameters. Furthermore, the shock variables \( \hat{\mu}, \hat{z}, \hat{\chi}, \hat{\psi}, e^R, e^R_k, \tilde{\sigma}_\omega \) are all set to zero.

Since the model has non-zero steady state inflation, we adjust the welfare loss function used to compare the performance of the policy rules to:

\[
\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ (\pi_t - \bar{\pi})^2 + \lambda \hat{Y}_t^2 \right]
\]

\( (A.8) \)

<table>
<thead>
<tr>
<th>Target</th>
<th>Data</th>
<th>Model</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \min \pi_t</td>
<td>_{ELB} )</td>
<td>1.5</td>
<td>1.486</td>
<td>( \hat{\lambda}_{f,L} )</td>
</tr>
<tr>
<td>( \min \hat{Y}_t )</td>
<td>-8.5</td>
<td>-8.49</td>
<td>( \hat{b}_L )</td>
<td>-0.105</td>
</tr>
<tr>
<td>( \mathbb{E}(ELB)</td>
<td>_{ELB} )</td>
<td>4</td>
<td>3.95</td>
<td>( \beta )</td>
</tr>
</tbody>
</table>

\( \tau \)

\( \pi \) 2.0 2.0

Table A.1: Calibration results. See text for details. \( \mathbb{E}(ELB)|_{ELB} \) is in quarters. Note: First three parameters are calibrated simultaneously to hit targets. Changing one of the three parameters will affect all three targets. Last two parameters have one-to-one relation with respective target. \( \pi \) is set directly.

\(^{43}\)The solution features 4 periods in regime 0, i.e. \( T = 5 \) and \( k_{32} = 1 \).
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Source</th>
<th>Standard Values</th>
<th>Optimal Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>TTR</td>
<td>Nakov (2008)</td>
<td>$\phi_{\pi} = 1.5$</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_y = 0.5$</td>
<td></td>
</tr>
<tr>
<td>TTR-1</td>
<td>Nakov (2008)</td>
<td>$\phi_{\pi} = 1.5$</td>
<td>$\phi_{\pi} = 1.14$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_y = 0.5$</td>
<td>$\phi_y = 0.28$</td>
</tr>
<tr>
<td>TTR+1</td>
<td>Nakov (2008)</td>
<td>$\phi_{\pi} = 1.5$</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_y = 0.5$</td>
<td></td>
</tr>
<tr>
<td>TTRS</td>
<td>Nakov (2008)</td>
<td>$\phi_{\pi} = 1.5$</td>
<td>$\phi_{\pi} = 51.50$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_y = 0.5$</td>
<td>$\phi_y = 48.77$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_i = 0.8$</td>
<td>$\phi_i = 0.98$</td>
</tr>
<tr>
<td>TTRP</td>
<td>Wolman (2005)</td>
<td>$\phi_p = 1.5$</td>
<td>$\phi_p = 1.23$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_y = 0.5$</td>
<td>$\phi_y = 1.23$</td>
</tr>
<tr>
<td>ATR</td>
<td>Reifschneider and Williams (2000)</td>
<td>$\alpha = 1$</td>
<td>$\alpha = 92.09$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_{\pi} = 1.5$</td>
<td>$\phi_{\pi} = 100$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi_y = 0.5$</td>
<td>$\phi_y = 34.23$</td>
</tr>
<tr>
<td>SUP</td>
<td>Rotemberg and Woodford (1999)</td>
<td>$\phi^{SUP} = 1.28$</td>
<td>$\phi^{SUP} = 1.48$</td>
</tr>
<tr>
<td>AIT</td>
<td>Reifschneider and Wilcox (2019)</td>
<td>$\phi^{AIT} = 5$</td>
<td>$\phi^{AIT} = 18.7$</td>
</tr>
</tbody>
</table>

Table A.2: Policy rules, standard parametrization, and optimal parametrization. The Table reports, for each Taylor-type policy rule, the parameter values used for the simulations. The second column reports values used in the literature as well as the source. The last column reports optimal values, that minimize the welfare loss (26). In TTR and TTR+1 the standard values are already optimal.
A.6 Robustness checks

A.6.1 Simple New Keynesian model with -0.5% inflation drop – Additional policy rules and impulse responses

Figure A.1: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.2: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.3: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
### Simple New Keynesian model with 0% inflation drop

**Welfare Loss**

\[ E_0[\tau + k_i - \tilde{T}] \]

**Volatility**

(1) (2) (3) (4) (5) (6) (7)

<table>
<thead>
<tr>
<th></th>
<th>1.000</th>
<th>1.000</th>
<th>1.000</th>
<th>1.000</th>
<th>1.000</th>
<th>1.000</th>
<th>1.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: baseline rules</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OCP</td>
<td>9.938\times 10^{-4}</td>
<td>15.471</td>
<td>6.409 \times 10^{-3}</td>
<td>5.932 \times 10^{-4}</td>
<td>1.438 \times 10^{-3}</td>
<td>-2.415</td>
<td>3.366</td>
</tr>
<tr>
<td>TTR</td>
<td>3.150</td>
<td>0.646</td>
<td>7.814</td>
<td>0.000</td>
<td>0.645</td>
<td>3.077</td>
<td>0.016</td>
</tr>
<tr>
<td>HD-NGDPT</td>
<td>1.577</td>
<td>1.108</td>
<td>3.613</td>
<td>0.203</td>
<td>1.098</td>
<td>1.832</td>
<td>0.499</td>
</tr>
<tr>
<td>SDTR</td>
<td>1.191</td>
<td>0.693</td>
<td>1.512</td>
<td>0.974</td>
<td>0.700</td>
<td>1.393</td>
<td>0.937</td>
</tr>
<tr>
<td>ATR</td>
<td>1.361</td>
<td>0.646</td>
<td>2.437</td>
<td>0.634</td>
<td>0.653</td>
<td>1.775</td>
<td>0.744</td>
</tr>
<tr>
<td>SUP</td>
<td>1.335</td>
<td>0.000</td>
<td>1.860</td>
<td>0.980</td>
<td>0.411</td>
<td>1.778</td>
<td>0.902</td>
</tr>
</tbody>
</table>

**Panel B: additional rules**

<table>
<thead>
<tr>
<th></th>
<th>3.213</th>
<th>0.646</th>
<th>7.971</th>
<th>0.000</th>
<th>0.645</th>
<th>3.105</th>
<th>0.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLT</td>
<td>3.213</td>
<td>0.646</td>
<td>7.971</td>
<td>0.000</td>
<td>0.645</td>
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<td>0.000</td>
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<tr>
<td>NGDPT</td>
<td>4.082</td>
<td>0.000</td>
<td>10.119</td>
<td>0.006</td>
<td>0.186</td>
<td>3.824</td>
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</tr>
<tr>
<td>TRP</td>
<td>1.469</td>
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<td>2.496</td>
<td>0.775</td>
<td>0.352</td>
<td>2.094</td>
<td>0.787</td>
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<td>TRS-1</td>
<td>1.622</td>
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<td>3.500</td>
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<td>0.641</td>
<td>2.243</td>
<td>0.560</td>
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<td>TR-1</td>
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<td>0.646</td>
<td>7.971</td>
<td>0.000</td>
<td>0.645</td>
<td>3.105</td>
<td>0.000</td>
</tr>
<tr>
<td>NY Fed Rule</td>
<td>67.154</td>
<td>0.782</td>
<td>124.781</td>
<td>28.239</td>
<td>1.700</td>
<td>12.431</td>
<td>-5.177</td>
</tr>
</tbody>
</table>

Table A.3: Some metrics for selected interest rate rules the simple two-equation NK model in the presence of a correlated cost push shock. All rows except the first show values normalized with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock and a correlated cost push shock such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). The model is calibrated with the standard EW (2003) parameter values reported in footnote 19.
Figure A.4: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.5: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.6: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.7: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
### A.6.3  Simple New Keynesian model with -2% inflation drop

<table>
<thead>
<tr>
<th></th>
<th>Welfare Loss</th>
<th>$E[\tau + k_T - T]$</th>
<th>Volatility $\chi$</th>
<th>Volatility $\pi$</th>
<th>Volatility $i$</th>
<th>Impact $\chi$</th>
<th>Impact $\pi$</th>
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<tbody>
<tr>
<td>OCP</td>
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<td>14.073</td>
<td>2.784 $10^{-3}$</td>
<td>2.422 $10^{-4}$</td>
<td>1.326 $10^{-3}$</td>
<td>-1.597</td>
<td>2.141</td>
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#### PANEL A: baseline rules

<table>
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<tr>
<th></th>
<th>Welfare Loss</th>
<th>$E[\tau + k_T - T]$</th>
<th>Volatility $\chi$</th>
<th>Volatility $\pi$</th>
<th>Volatility $i$</th>
<th>Impact $\chi$</th>
<th>Impact $\pi$</th>
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<tbody>
<tr>
<td>OCP</td>
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<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>TTR</td>
<td>7.975</td>
<td>0.711</td>
<td>17.869</td>
<td>0.865</td>
<td>0.699</td>
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<tr>
<td>HD-NGDPT</td>
<td>1.537</td>
<td>1.096</td>
<td>3.372</td>
<td>0.217</td>
<td>1.083</td>
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<tr>
<td>SDTR</td>
<td>1.202</td>
<td>0.768</td>
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<td>0.774</td>
<td>1.422</td>
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<tr>
<td>ATR</td>
<td>1.647</td>
<td>0.711</td>
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<td>0.383</td>
<td>0.714</td>
<td>2.133</td>
<td>0.556</td>
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<tr>
<td>SUP</td>
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<td>0.000</td>
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<td>1.997</td>
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#### PANEL B: additional rules

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<th>$E[\tau + k_T - T]$</th>
<th>Volatility $\chi$</th>
<th>Volatility $\pi$</th>
<th>Volatility $i$</th>
<th>Impact $\chi$</th>
<th>Impact $\pi$</th>
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</thead>
<tbody>
<tr>
<td>PLT</td>
<td>3.660</td>
<td>0.711</td>
<td>8.745</td>
<td>0.006</td>
<td>0.699</td>
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<tr>
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<td>0.711</td>
<td>8.377</td>
<td>0.002</td>
<td>0.699</td>
<td>3.229</td>
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<td>0.017</td>
<td>0.242</td>
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<tr>
<td>TTRS-1</td>
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<td>0.000</td>
<td>3.523</td>
<td>0.555</td>
<td>0.403</td>
<td>2.603</td>
<td>0.623</td>
</tr>
<tr>
<td>TTR-1</td>
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<td>7.976</td>
<td>0.001</td>
<td>0.695</td>
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<td>AIT</td>
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<td>0.711</td>
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<td>0.700</td>
<td>3.327</td>
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<td>NY Fed Rule</td>
<td>15.284</td>
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<td>30.660</td>
<td>4.234</td>
<td>1.157</td>
<td>5.849</td>
<td>-1.549</td>
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Table A.4: Some metrics for selected interest rate rules the simple two-equation NK model in the presence of a correlated cost push shock. All rows except the first show values normalized with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock and a correlated cost push shock such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). The model is calibrated with the standard EW (2003) parameter values reported in footnote 19.
Figure A.8: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% constant under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.9: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.10: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.11: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
### A.6.4 Simple New Keynesian model with -0.5% inflation drop – Price rigidity

<table>
<thead>
<tr>
<th></th>
<th>Welfare Loss ((1))</th>
<th>(\mathbb{E}<em>0[\tau + k</em>{\pi} - \bar{T}]) ((2))</th>
<th>Volatility (x) ((3))</th>
<th>Volatility (\pi) ((4))</th>
<th>Volatility (i) ((5))</th>
<th>Impact (x) ((6))</th>
<th>Impact (\pi) ((7))</th>
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</thead>
<tbody>
<tr>
<td>OCP</td>
<td>4.896 (10^{-4})</td>
<td>16.549</td>
<td>7.661 (10^{-3})</td>
<td>1.081 (10^{-5})</td>
<td>1.524 (10^{-3})</td>
<td>(-2.755)</td>
<td>(0.072)</td>
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</table>

**Panel A: baseline rules**

<table>
<thead>
<tr>
<th>Rule</th>
<th>(\mathbb{E}<em>0[\tau + k</em>{\pi} - \bar{T}])</th>
<th>Volatility (x)</th>
<th>Volatility (\pi)</th>
<th>Volatility (i)</th>
<th>Impact (x)</th>
<th>Impact (\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCP</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>TTR</td>
<td>6.550</td>
<td>0.604</td>
<td>6.668</td>
<td>1.312</td>
<td>0.608</td>
<td>2.722</td>
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<tr>
<td>HD-NGDPT</td>
<td>1.137</td>
<td>1.048</td>
<td>1.162</td>
<td>0.034</td>
<td>1.036</td>
<td>1.116</td>
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<td>SDTR</td>
<td>1.085</td>
<td>1.020</td>
<td>1.108</td>
<td>0.079</td>
<td>1.018</td>
<td>1.102</td>
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<tr>
<td>ATR</td>
<td>1.876</td>
<td>0.794</td>
<td>1.916</td>
<td>0.105</td>
<td>0.824</td>
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<td>SUP</td>
<td>4.520</td>
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<td>4.604</td>
<td>0.797</td>
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**Panel B: additional rules**

<table>
<thead>
<tr>
<th>Rule</th>
<th>(\mathbb{E}<em>0[\tau + k</em>{\pi} - \bar{T}])</th>
<th>Volatility (x)</th>
<th>Volatility (\pi)</th>
<th>Volatility (i)</th>
<th>Impact (x)</th>
<th>Impact (\pi)</th>
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<tr>
<td>PLT</td>
<td>6.328</td>
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<td>6.445</td>
<td>1.133</td>
<td>0.608</td>
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<td>NGDPT</td>
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<td>4.584</td>
<td>0.439</td>
<td>0.608</td>
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<tr>
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<td>0.183</td>
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<tr>
<td>TTRS-1</td>
<td>1.522</td>
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<td>TTR-1</td>
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<td>AFT</td>
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<td>0.911</td>
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<td>23.399</td>
<td>32.647</td>
<td>1.449</td>
<td>4.404</td>
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</tbody>
</table>

*Table A.5:* Some metrics for selected interest rate rules the simple two-equation NK model with increased price rigidity (lower \(\kappa\)). All rows except the first show values normalized with respect to the optimal commitment policy (OCP; first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). Remaining parameters are calibrated according to standard EW (2003) values reported in footnote 19.
Figure A.12: Dynamic response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower \( \kappa \)), under different policy rules. Shocks are calibrated such that output falls by 7.5\% and inflation by 0.5\% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. Remaining parameters are calibrated according to standard EW (2003) values reported in footnote 19.
Figure A.13: Dynamic response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower $\kappa$), under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The list of acronyms is detailed in Table 1. Remaining parameters are calibrated according to standard EW (2003) values reported in footnote 19.
Figure A.14: Average impulse response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower $\kappa$), under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
Figure A.15: Average impulse response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower \( \kappa \)), under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The list of acronyms is detailed in Table 1. The parametrization is reported in Table A.2.
### A.6.5 Simple New Keynesian model with -0.5% inflation drop – Optimized policy rules

<table>
<thead>
<tr>
<th>Welfare Loss</th>
<th>(E_0[\tau + k_T - \bar{T}])</th>
<th>Volatility (x)</th>
<th>Volatility (\pi)</th>
<th>Volatility (i)</th>
<th>Impact (x)</th>
<th>Impact (\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCP</td>
<td>8.252 (10^{-4})</td>
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<td>5.356 (10^{-3})</td>
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<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
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<tr>
<td>TTR</td>
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<td>3.563</td>
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<td>1.094</td>
<td>1.818</td>
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<td>1.514</td>
<td>0.975</td>
<td>0.716</td>
<td>1.400</td>
</tr>
<tr>
<td>ATR</td>
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<td>1.070</td>
<td>1.237</td>
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<td>8.058</td>
<td>0.000</td>
<td>0.657</td>
<td>3.133</td>
</tr>
<tr>
<td>TTRS-1</td>
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<td>0.651</td>
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<td>8.079</td>
<td>0.000</td>
<td>0.657</td>
<td>3.138</td>
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</table>

**Panel A:** baseline rules

**Panel B:** additional rules

Table A.6: Some metrics for selected interest rate rules the simple two-equation NK model with optimized policy rules. All rows except the first show values normalized with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). The model is calibrated with the standard EW (2003) parameter values reported in footnote 19.

### A.6.6 Simple New Keynesian model – Fixed-Length Forward Guidance Experiment – Contingen-cies
Figure A.16: Optimal Commitment Policy (Panel a, left) and Optimal Fixed-Length Forward Guidance (Panel b, right) in a two-state Markov process. The size of the Markov shock is calibrated to achieve a drop in output in the initial period of $-7.5\%$ under discretion. The purple solid line reports the IRF under the optimal policy under commitment (Ramsey plan). The purple red line reports the IRF under the optimal fixed length forward guidance. The gray solid lines report the evolutions under single contingencies (contingencies 15 to 25 are reported). The vertical axes for inflation and output report deviations from steady state, in percentage points (annualized figures). The vertical axis for the interest rate reports annualized percentage points.
Figure A.17: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under baseline policy rules. Colored lines show paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$). The two-state Markov shocks switch to low state in Q4-07 and revert to the absorbing state after 32 quarters (32nd contingency). The vertical axes report deviations from detrended steady state, in percentage points (annualized figures). The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1. FRBNY rule refers to Equation (A.7).
Figure A.18: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under additional policy rules. Colored lines show paths for output ($\hat{Y}_t$), inflation ($\pi$), the nominal interest rate ($i$), and nominal GDP ($\hat{N}$). Dotted red line is data. The two-state Markov shocks switches to low state in Q4-07 and reverts to the absorbing state after 32 quarters (32nd contingency). The vertical axes for $\hat{Y}_t$ and $\hat{N}$ report deviations from detrended steady state, in percentage points (annualized figures). The vertical axes for $\pi$ and $i$ report annualized percentage points. The horizontal axis shows quarter and calendar year. See Section A.3 for details on data and Section A.5.2 for calibration. The list of acronyms is detailed in Table 1.
Figure A.19: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under additional policy rules. Colored lines show paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$). The two-state Markov shocks switch to low state in Q4-07 and revert to the absorbing state after 32 quarters (32nd contingency). The vertical axes report deviations from detrended steady state, in percentage points (annualized figures). The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1.
Figure A.20: Average impulse response to a preference shock and a correlated cost push shock in FRBNY model, under baseline policy rules. The vertical axes report deviations from detrended steady state, in percentage points (annualized figures). The vertical axes for $\pi$ and $i$ report annualized percentage points. The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1. FRBNY rule refers to Equation (A.7).
Figure A.21: Average impulse response to a preference shock and a correlated cost push shock in FRBNY model, under additional policy rules. The vertical axes report deviations from detrended steady state, in percentage points (annualized figures). The vertical axes for \( \pi \) and \( i \) report annualized percentage points. The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1.